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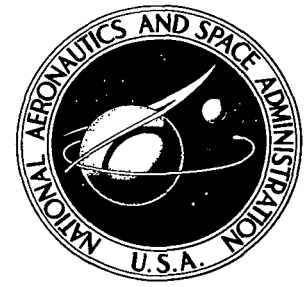

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NUMERICAL METHODS FOR SOLVING PROBLEMS OF MECHANICS OF CONTINUOUS MEDIA

Edited by O. M. Belotserkovskiy

Computer Center of the USSR Academy of Sciences
Moscow, 1969

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TABLE OF CONTENTS

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O. M. Belotserkovskiy

INTRODUCTION

Present practical needs require that applied scientists cope with various types of problems which can, in most cases, be successfully solved with the required accuracy solely by numerical methods using electronic computers. This obviously does not mean that analytical methods which permit solutions in "closed" form will not be further developed; however, it is entirely clear that the class of problems which may be solved in this manner is quite small and it is hence important to develop general numerical algorithms for the study of problems of mathematical physics. This appears to be particularly timely in the mechanics of continuous media (gasdynamics, theory of elasticity, etc.), which is due to a number of circumstances. /3*

1. Difficulties in carrying out experiments. Experiments concerning phenomena attendant, for example, to hypersonic flight velocities are accompanied by high temperatures, which result in dissociation and ionization and in a number of cases also in gas "glow." In these cases the modeling of the phenomenon under laboratory conditions is extremely difficult, since similitude between the full-scale specimen and the experimental model can no longer be obtained by satisfying the classical similitude criteria — equality of Mach and Reynolds numbers of the model and prototype. It is also required that absolute pressures and absolute temperatures be equal, which is possible only when the dimensions of the model and prototype are the same. All this shows the great technical difficulties and expense of experimental studies, without mentioning the fact that in many cases the experimental data are of highly limited nature.

However, the above is not intended to degrade the importance of experiments. They will always remain the cornerstone of studies which verify (or disprove) the model and solution of a given theoretical approach.

2. Complexity of the equations used. The extensive use of numerical methods in the mechanics of continuous media also follows from the fact that the equations of aerodynamics, gasdynamics and the theory of elasticity constitute a most complex (compared with other fields of mathematical physics) system of partial differential equations. /4

In general, this is a mixed-type nonlinear system with an unknown shape of the transition surface (where the type of equation changes) and with "moving boundaries," i.e., the boundary conditions of the problem are specified at surfaces or lines which themselves are subject to determination during computations.

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*Numbers in the margin indicate pagination in the foreign text.

Here the range of variation of starting functions is so wide that ordinary methods of analytic studies (linearization of equations, series expansion, the small-parameter method, etc.) are in general not suitable here for obtaining a complete solution of the problem.

It should be noted that preliminary analytical solutions of the various properties of the problem may be of great help in formulating large complex problems with computers, and sometimes are simply decisive for successful realization of the numerical algorithm.

In the final analysis, the success of using a given algorithm with the smallest expenditure of machine time depends on the smoothness of the functions used. Hence the selection of independent variables, different forms of writing the starting system of equations (which may be mathematically equivalent, but not equivalent with respect to their approximate representation), the use of the exact integrals of the system, determination of directions along which the functions are represented, the structure of computational grids — all play an important role in working out the numerical algorithm.

3. We shall now consider still another feature of algorithms used in solving specific problems of mechanics of continuous media. At present, as is known, numerical methods are coming into permanent use in investigations of design offices and research institutes. The numerous successes in the study of space, in the practice of optimal control, selection of efficient aircraft shapes, etc., owe a great deal to serial computations and the use of information thus obtained. The volume of information provided by a correctly stated, well modeled and efficiently algorithmed problem is much more complete and substantially cheaper than corresponding experiments. However, extensive practical use of numerical methods requires that these be sufficiently simple and reliable. /5

Thus, on the one hand, one has to deal here with very complicated mathematical problems, while on the other it is necessary to develop sufficiently simple and reliable numerical methods which can be used for serial computations under conditions prevailing in research institutes and design offices.

This collection is primarily concerned with numerical methods of solving gasdynamic problems. In this sense the content of this series of articles is somewhat more narrow than the collection's title.

We note first not only that no mathematical theorems of existence and uniqueness are proved for the overwhelming majority of gasdynamic problems, but that frequently it is not sure whether such theorems can be obtained at all. The mathematical statement proper of the problem is, as a rule, not formulated within the rigorous meaning of this term, and only the physical statement is given, which is not the same thing. The mathematical difficulties in the study of problems of these types are due to nonlinearity of equations and also to the large number of independent variables.

The situation is similar with respect to methods for solving gasdynamic equations. Studies pertaining to the feasibility of implementing the algorithm, its convergence to the sought solution and stability have at present been carried

out rigorously only for linear systems, and in a number of cases only for equations with constant coefficients. Being nevertheless faced with the need to solve the problem, the computational mathematician is forced to use algorithms already known and to even a greater extent develop new methods the applicability of which does not have a rigorous mathematical proof. It should not be thought that the situation of such a mathematician differs much from that of any investigator in a new field. One can find a large number of examples in sciences, including mathematics, when new ideas and concepts arose and were successfully used without a rigorous substantiation, which appeared later. Obviously, this does not imply that new computational algorithms can be developed blindly, /6
without regard for a clear statement of the problem and without delving deeply into its physical meaning. This approach inevitably results in numerous mistakes; with the loss of time and what is most important, experience, which was obtained without a theoretical substantiation, not providing any basis for the subsequent development of the method.

Attention is called here to this generally simple problem for the reason that it has hitherto been considered by some that the main thing is to write the differential equations, and the rest then reduces to a trivial substitution of finite differences for the derivatives and to programming, which is frequently given more than its deserved importance. In conjunction with this it is useful to formulate the principal stages in the numerical solution of mechanics of physics problems on an electronic computer as follows: 1) construction of the physical model and mathematical statement of the problem; 2) development of a computational algorithm and its theoretical investigation; 3) programming (manual or automatic) and formal debugging of the program; 4) procedural debugging of algorithm — checking its performance with specific problems; elimination of shortcomings and experimental study of the algorithm; 5) serial computations, accumulation of experience, estimating the effectiveness and limits of applicability of the algorithm.

Mathematical theory, physical and numerical experiments using the computer are used together and consistently at all stages. How this is done at each stage is best illustrated by solving specific problems, which shall be done below. Hence only a few general remarks are in order here.

The basic principle of using mathematical results consists in the fact that conditions allowing one to solve a problem in the simpler and more particular cases should be satisfied also for more general and complicated cases. At the same time the consideration of the physics of the phenomenon provides a qualitative picture which is used for checking and refining the statement of the problem. Finally, the final experimental check makes it possible to determine the correctness of the assumptions made and to give an estimate of the algorithm and of the solution, in particular of the latter's accuracy. It should be noted here that the accuracy of the numerical solution of a formulated differential problem should be checked purely mathematically, without resort to physical experimental data. The latter can used for qualitative comparisons, while quantitative /7
comparison of calculations with the experiment should provide information on the extent to which the assumed physical model approaches actual conditions.

The lecture series "Numerical Methods for Solving Problems of Mechanics of Continuous Media" consists of a number of communications illuminating different approaches in this field. Analysis is based chiefly on specific examples.

In a number of cases the topics of the papers are similar and it is possible that the same problem is presented by the lectures from different points of view. This is natural, since one of the tasks of the summer session was to familiarize the participants with different aspects and points of view which arose over a wide range of research with the advent of electronic computers and made it in general possible to obtain sufficiently accurate solution of complete equations of gasdynamics, meteorology, etc.

The first to be presented are finite-difference methods, and these are followed by schemes of the numerical method of integral relations and by the method of characteristics. This subdivision of methods into the above groups is quite arbitrary, since many finite-difference schemes require satisfaction of integral relations (conservation laws), while schemes involving the use of characteristics are substantially difference methods using special characteristic grids.

V. V. Rusanov and V. F. D'yachenko present in their lectures finite-difference methods for solving unsteady gasdynamic problems (stabilization methods, methods of "through" computation, problem of breakup of an arbitrary discontinuity, etc.). G. I. Marchuk considers difference schemes of the "decomposition" method and touches upon questions of its application in problems of dynamic meteorology and hydrodynamics. The use of the small-parameter method in constructing numerical solutions of equations of mathematical physics is considered in the lecture by A. A. Dorodnitsyn. O. M. Belotserkovskiy presents numerical methods for solving steady-state gasdynamics problems (method of integral relations and lines, numerical method of characteristics, schemes with pseudo-viscosity and their applications). V. V. Shchennikov considers schemes of the method of integral relations and of the finite-difference method as applied to calculation of viscous boundary layer flows.

The selection of problems used to illustrate the numerical methods is obviously related to the interests of the lecturers and in no way pretends to fully describe the application of the given method.

Despite the fact that this collection is published quite some time after the session was held, it was decided not to supplement the collection in any way, since this would only delay its publication. /8

November, 1969.

O. M. Belotserkovskiy
V. V. Rusanov

DIFFERENCE METHODS IN SOLVING PROBLEMS OF GASDYNAMICS

V. V. Rusanov

1. Some Information from Gasdynamics

Let us consider the flow of gas occurring in some region in space. Let r be the radius vector of the point and t the time. The principal flow variables of the gas: velocity U, pressure p, density ρ and others are defined for a gas particle* located at the given point at the given time and are functions of r and t. The gas flow is called steady if the functions governing it are independent of time, i.e., if the state of the gas in a given point in space does not change with time. In the opposite case the flow is called unsteady. /9

Let f(r, t) be some parameter of the gas. We can follow its change with time in two ways — either at a given point with r = const, or for a given particle, in which case r = R(t), where R(t) describes the path traversed by the particle. The change in the first case is defined by the derivative $\partial f/\partial t|_{r=\text{const}}$. In the second case f(r, t) = f(R(t), t) and $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial R}\cdot\frac{dR}{dt}$, this expression is called the total derivative.

If f is a scalar function, then $\partial f/\partial R = \text{grad}\, f$. If if is a vector, then $\partial f/\partial R$ is a tensor.

To find dR/dt we note that r = R(t) is the equation of the path of a particle which at time t in point r has a velocity U(r, t). Hence dR/dt = U and the expression for the total derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + V\,\text{grad}\, f.$$

In addition to the pressure and density we will have to deal with the following functions: T - temperature, ε - internal energy of unit mass consisting of the total energy of the molecules, h - enthalpy and S - entropy. Only two of the above quantities are independent. We shall assume that these are pressure p and density ρ and shall treat the other quantities as functions thereof. /10

*A "gas particle" is an arbitrary concept, denoting a small gas volume within which all the flow variables can be regarded as constant, but which still contains enough molecules so that it can be treated as a continuum.

The equations of motion of a gas can be obtained by applying to an individual gas particle the basic laws of physics and mechanics — laws of conservation of mass, momentum and energy, with allowance for thermodynamic processes occurring in the particle. We consider the equilibrium flow of a gas, i.e., a flow such that during any given time the interaction between gas particles is of purely mechanical nature.

The gasdynamic equations in this case have the form: $\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{U} = 0$ - equation of conservation of mass; $\frac{d\mathbf{V}}{dt} + \frac{1}{\rho} \operatorname{grad} p = 0$ - equation of conservation of momentum, and $\rho \frac{d\varepsilon}{dt} + p \operatorname{div} \mathbf{U} = 0$ - equation of conservation of energy.

Combining the first and third equations, we get

$$\frac{d\varepsilon}{dt} + p \frac{d\left(\frac{1}{\rho}\right)}{dt} = 0 \quad \text{or} \quad T \frac{dS}{dt} = 0.$$

Expression $dS/dt = 0$ means that the entropy is constant along the particle path, and in steady flow it is constant along the streamline. Finally, the last equation can be transformed to the following form, which shall be subsequently used:

$$\frac{dp}{dt} + \rho c^2 \operatorname{div} \mathbf{U} = 0,$$

where

$$c^2 = \frac{p - \rho^2 \frac{\partial \varepsilon}{\partial \rho}}{\rho^2 \frac{\partial \varepsilon}{\partial p}} = -\frac{\frac{\partial S}{\partial \rho}}{\frac{\partial S}{\partial p}} = \left(\frac{dp}{d\rho}\right)_{S = \text{const}}.$$

Quantity c^2 is a function of p and ρ and, as is known from acoustics, is the square of the speed of sound. /11

For an ideal gas with constant ratio of specific heats $\gamma = c_p/c_v$ we have

$$c^2 = \gamma \frac{p}{\rho}; \qquad h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}; \qquad \varepsilon = \frac{1}{\gamma - 1} \frac{p}{\rho}.$$

Both steady and unsteady problems of gasdynamics can be subdivided into one-, two- and three-dimensional, depending on the number of space coordinates on which the gas flow substantially depends. We shall basically consider problems

with three variables, i.e., we shall consider unsteady two-dimensional and steady three-dimensional problems.

In addition to differential equations, we will have to deal with boundary conditions. In problems to be considered one encounters two types of boundary surfaces — impenetrable surface of a body and the shock wave. A boundary condition at the surface of a body is simply the condition of impermeability, i.e., requirement that the velocity component normal to the body be zero.

The boundary conditions at a shock wave are somewhat more complicated. Let us examine the surface of a shock which is a surface of discontinuity of gasdynamic functions. Let ν be the normal to the shock wave and D the absolute shock wave velocity in space in the direction of this normal (the velocity with which the wave surface moves in space along the normal); then νD is the vector velocity of the wave. Both ν and D are functions of a point on the wave surface. The gas velocity relative to the wave is expressed by the equation

$$V = U - \nu D.$$

Let f_+ and f_- be the values of some function f on the two sides of the shock wave. We denote the jump in function f across the wave by [f]:

$$[f] = f_+ - f_-.$$

We note that the velocity V of the gas relative to the wave as well as its velocity U in space are different on both sides of the shock wave, while the velocity νD of the wave proper in space is naturally the same on both sides of the wave. The boundary conditions at the shock wave are thus written as:

/12

$$[\rho V_\nu] = 0; \quad [U_\tau] = 0; \quad [p + \rho V_\nu^2] = 0; \quad \left[h + \frac{V_\nu^2}{2}\right] = 0,$$

where V_ν is the projection of V on the normal, $V_\nu = U_\nu - D$, and U_τ is the velocity component in the plane tangent to the shock surface. For steady flow D = 0, i.e., the shock wave is stationary in space.

Some remarks now on the characteristics of gasdynamic equations. The concept of characteristics of partial differential equations has many aspects. From the point of view of the theory of differential equations a surface is called a characteristic if under the initial conditions on it the Cauchy problem is unsolvable. Another property of a characteristic surface is the fact that some differential relation between the sought functions is satisfied in each of its points; here all the differentiations are made along directions lying in the plane tangent to the surface. Both these properties are closely interrelated and can serve as points of departure for determining the characteristic surface.

It can also be shown that characteristics are surfaces of weak discontinuities, i.e., discontinuities of derivatives in the solution, and surfaces along which small perturbations are propagated through the gas. This latter property will be of primary interest here.

If we examine all the characteristic surfaces passing through one point, their envelope will form some surface similar to a deformed cone and termed the characteristic conoid. This conoid has the property that a small perturbation originating at a point propagates over a region bounded by the inner surface of the characteristic conoid with a vertex at this point.

The conical surface tangent to the characteristic conoid and having a common vortex with it is called the characteristic cone. The characteristic cone permits a very graphic physical and geometric interpretation in the case of three independent variables, i.e., in the case of unsteady two-dimensional and steady three-dimensional flows.

Let us, for example, examine the two-dimensional unsteady flow in the (x, y) surface, the third coordinate being the time t. It is assumed for simplicity that all the quantities change little in the vicinity of the point under study; in particular, it is assumed that the gas velocity and the speed of sound can be regarded as constant over a small time interval. Let this velocity in the (x, y) plane be represented by the vector U, and let at some time t = 0 a small disturbance, from which a sonic wave propagates with velocity c, initiate in the coordinate origin.

During the time t the particle moves over segment Ut, while the disturbance propagates to all sides of it through a distance ct, and at time t the disturbed region will have a circular shape. Treating t as a parameter, we will get in the (r, t) space some cone, the section of which by plane t = const is a circle with a center at point Ut and radius ct (Fig. 1). This is the characteristic cone for a two-dimensional unsteady flow.

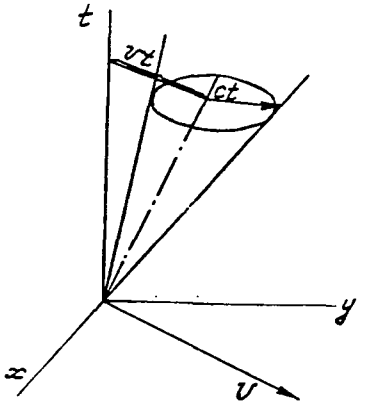

Figure 1.

In this case the characteristic cone will always exist, irrespective of the magnitude of velocities U and c. This is due to the fact that unsteady gasdynamic equations are always hyperbolic, i.e., a real characteristic cone always exists.

The perturbation is maintained also in the particle in which it originates; hence still another line will exist inside the cone, i.e., the path of the particle, along which the perturbation will also move. This line is a degenerate characteristic cone.

We also note that the characteristic cone does not touch and does not intersect the plane t = const, i.e., it moves out of it. This circumstance ensures that the Cauchy problem for the equations under study with initial conditions at $t = t_0$ always has a solution, i.e., these surfaces (of plane t = const) are not characteristic.

At any instant in time the perturbation originating at some point propagates only over a finite distance and conversely, for any point P, lying on the plane $t > t_0$ all the points in the plane $t = t_0$ which can affect it lie inside the base of the characteristic cone with vertex at point P. The dependence domain of point P is bounded.

If we take three space variables, then the situation will be similar. The characteristic cone, or hypercone, never intersects the space t = const. Hence any surface lying outside the characteristic cone and not intersecting it is said to be a three-dimensional type surface. /14

This three-dimensional type surface has the property that the Cauchy problem with the initial conditions imposed on it has a solution at some close proximity to the surface.

In addition to surfaces t = const, surfaces f(x, y) = const may also be surfaces of three-dimensional type. The Cauchy problem with initial data at f(x, y) = = const physically means that the initial conditions are specified at some stationary line.

A surface f(x, y) = const can be a three-dimensional type surface only in the case of $U > c$. If, however, $U < c$, then the characteristic cone will include within it the time axis t, the surface f(x, y) = const will always intersect it and, consequently, it will no longer be a three-dimensional type surface.

When $U > c$ the cone for steady three-dimensional flow exists and is real and the equations are hyperbolic. This is the case of supersonic flow. When $U < c$ the flow is subsonic and the steady-state equations are elliptical.

A similar situation also exists for three-dimensional steady-state flows. In this case the term three-dimensional type surface is also applied to any surface not intersecting the characteristic cone (if the latter is real).

2. Supersonic Gas Flow Around a Blunt Body. Statement of the Problem*

Let us consider an arbitrary smooth body placed in a supersonic gas flow, uniform and steady at infinity, with velocity U_∞, pressure p_∞ and density ρ_∞. Determination of the flow about the body reduces to finding a solution to the gasdynamic equation, satisfying the condition of impermeability at the body's surface and taking the specified values at infinity. /15 However, the problem as here stated does not a priori have a unique solution [4] and at present there is no theoretical basis for selecting the required solution. Hence use must be made of existing experimental data and one must postulate certain qualitative properties of the solution beforehand. Thus, it is well known that a shock wave is produced ahead of the forward part of a blunt body placed in a steady supersonic flow and that this wave separates the flow region adjoining the body from the undisturbed flow. Here the flow immediately behind the wave will be a priori known to be subsonic at points where the angle made by the wave surface with the vector of the undisturbed flow velocity U_∞ is larger than some value which depends on the Mach number M_∞.

If the body is bounded in a direction perpendicular to U_∞, then sufficiently far downstream the flow will again become supersonic and a transition surface will terminate the subsonic region.

Postulation of the flow pattern sharply reduces the class of permissible solutions and one may expect uniqueness of the solution, if such exists. Obviously, the latter will not always be true and there arises the problem of describing the class of bodies for which the above flow pattern is actually obtained.

From the mathematical point of view this question is just as complicated as the solution of the problem of supersonic flow past bodies in its general statement and at present no rigorous results are available on this point. Hence one must turn to experimental results and consider first the body shapes close to those for which the above flow pattern is actually obtained.

Along with the results of physical experiments, the existence of a solution can also be inferred from those of numerical experiments. In fact, if we obtain a numerical solution for some body which would correspond to the postulated flow pattern, then this serves as a weighty proof of the existence of a solution of the differential equations.

With the above observations in mind, we finally formulate the problem of flow around a body as follows. For a given body we shall seek the solution of gasdynamic equations in the region between the shock wave and the body on the assumption that no singularities exist in this region. The boundary condition at /16 the body's surface is that of its impermeability. The boundary conditions at the

*The most of the results presented in this and subsequent sections were published previously in [1-3].

shock wave, the location of which is to be determined, are the standard relationships at the discontinuity surface.

If the dimensions of the body are finite, then the solution of the kind sought does not a priori exist in its entire vicinity. Someplace in the afterbody its continuity may be disturbed (as a rule, a second "tail" shock wave is observed in experiments). However, these features do not affect the supersonic and subsonic regions farther downstream, where the solution can be found independently.

Here the overall problem naturally decomposes into two parts. We separate the space between the wave and the body into two parts, I and II, by a surface Π in such a manner that it lies in the supersonic region and is of the three-dimensional type (Fig. 2). Then the first problem will be determination of the flow in the bow part, in region I, where the equations are of the mixed type. The second problem is determination of the flow in the purely supersonic region II, where the equations are hyperbolic.

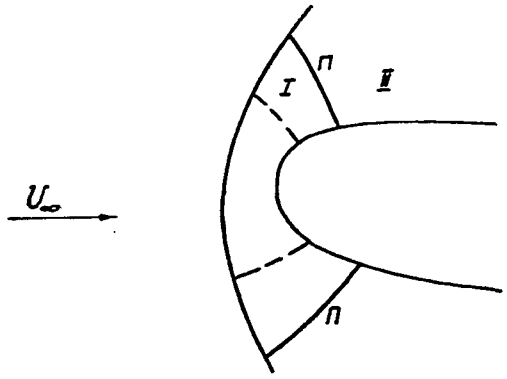

Figure 2.

We shall first consider the solution of the second problem, assuming that the first has been solved and the values of all the functions at π are known. Then in region II we will get a mixed problem with a free boundary for hyperbolic equations with three independent variables. If it were possible to construct an algorithm which would be used for finding the sought functions at a surface Π_1, situated farther downstream, from their values at Π, then the process could be repeated and the solution in region II could be constructed as long as the the postulated flow pattern is maintained.

Up to now all the deliberations have been of a general nature. Now it is expedient to make them more precise, introducing a certain coordinate system and writing equations and boundary conditions within this system.

We start with the cylindrical coordinates (z, r, φ) whose axis is drawn inside the body in such a manner that it has one common point with it. Let u, v and w be the components of the velocity vector $\overline{U}$ along the coordinate axes. The system

of steady equations of gasdynamics is written in matrix form

$$\Upsilon \frac{\partial X}{\partial t} + \Phi \frac{\partial X}{\partial r} + \Omega \frac{\partial X}{\partial \varphi} + \Lambda = 0,$$

where $X = \begin{Bmatrix} u \\ v \\ w \\ p \\ \rho \end{Bmatrix}$ and Υ, Φ, and Ω are fifth-order matrices, functions of the components of X.

For example,

$$\Upsilon = \begin{Bmatrix} u & 0 & 0 & \rho^{-1} & 0 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ \rho c^2 & 0 & 0 & u & 0 \\ \rho & 0 & 0 & 0 & u \end{Bmatrix}$$

etc.

Although it is easy to write equations in cylindrical coordinates, it is inconvenient to use them for solving the mixed problem, since the region has an unknown boundary. Hence we replace the independent variables, replacing (z, r, φ) by coordinates (ξ, η, ϑ), so that the coordinate surface $\xi = 0$ and $\xi = 1$ would be respectively the surface of the body and of the shock wave. One way of doing so is the following. We mark out for each η on the z axis a point A with coordiante $z = \zeta(\eta)$ and draw a cone with apex half-angle $\omega(\eta)$ with a vertex in this point (Fig. 3).

Let the cone's generatrix intersect the surfaces of the body and the shock wave at points B and C, respectively in the plane $\varphi = \vartheta = \text{const}$. Let P be any point lying on segment BC. We denote

$$AB = G(\eta, \vartheta); \quad AC = F(\eta, \vartheta)$$

and /18

$$\xi = \frac{BP}{BC} = \frac{AP - G}{F - G}.$$

Then it is easy to see that

$$\begin{aligned} z &= \zeta(\eta) - \{G(\eta,\vartheta) + \xi[F(\eta,\vartheta) - G(\eta,\vartheta)]\} \cos\omega(\eta); \\ r &= \{G(\eta,\vartheta) + \xi[F(\eta,\vartheta) - G(\eta,\vartheta)]\} \sin\omega(\eta); \\ \varphi &= \vartheta \end{aligned}$$

is precisely the sought substitution of variables.

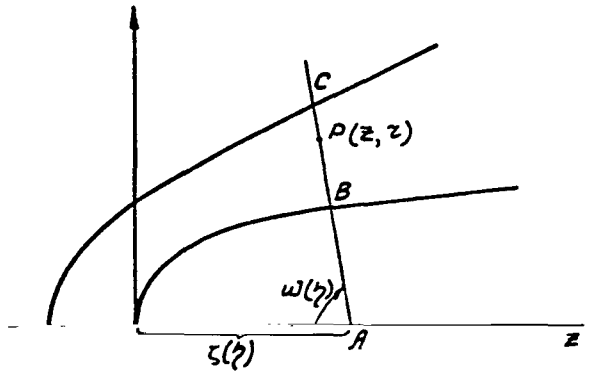

Figure 3.

If surfaces η = const are of the three-dimensional type, then they can be used as a system of surfaces II and mixed problem in the region with fixed boundaries

$$\eta \geq \eta_0; \qquad 0 \leq \xi \leq 1; \qquad 0 \leq \vartheta \leq 2\pi$$

can be considered for the system

$$A\frac{\partial X}{\partial \xi} + B\frac{\partial X}{\partial \eta} + C\frac{\partial X}{\partial \vartheta} + \Gamma = 0,$$

where A, B and C are matrices which are functions of X and of the derivatives of z and r with respect to ξ, η and ϑ. The system is assumed to be η-hyperbolic and the initial conditions are specified at $\eta = \eta_0$. The boundary conditions are specified at $\xi = 0$ and $\xi = 1$. The coefficients of this transformed system contain derivatives of the new coordinates with respect to the old, including also derivatives of the sought function F, defining the wave shape.

We have thus reduced the finding of the solution in region II to the mixed problem for a hyperbolic system of equations in a region of very simple shape. At all the points of this region we seek the values of components of vector X, while at points of the boundary $\xi = 1$ we seek the values of the function F for which the initial values are known at the segment $\xi = 1$, $\eta = \eta_0$, $0 \leq \vartheta \leq 2\pi$. The differential equation needed for finding function F is contained in the boundary conditions at the wave. In fact, the boundary conditions contain function F proper, as well as its derivatives F_η and F_ϑ. Upon eliminating F_η from them, we get four conditions relating the components of vector X and a fifth condition from which the value of F_η is determined. /19

We now examine the problem in region I. It is bounded by the shock wave, the surface of the body and the transition surface or, more precisely, the limiting

characteristic surface.* The difference in the type of equations in regions I and II is responsible for the substantially different statements of the problems. In the case of supersonic flow, when the equations are hyperbolic, the calculations can be carried out in sequence from the initial surface downstream, which appreciably simplifies the solution. In the case of mixed-type equations, the disturbances propagate from each point in all directions and the problem has to be solved simultaneously for the entire region. A very effective way for solving these problems is the so-called stabilization method. It is based on the physical fact that under actual conditions the flow about the body always arises as the limit of the unsteady flow in a sufficiently prolonged motion of a body with constant velocity and with constant fluid parameters. It therefore should be expected that for constant boundary conditions at the body and at infinity, the solution of the problem of unsteady flow past a body will approach the solution of the steady problem as time approaches infinity, irrespective of the initial flow. Since the equations of unsteady gas flow are always hyperbolic, the problem reduces to solving the mixed problem for a hyperbolic system of equations. With reference to the qualitative flow pattern, the boundary conditions for the problem at hand should be specified at the surface of the body and at the shock wave, the location /20
of the latter not being a priori known. Instead of the characteristic surface it is more expedient to consider the previously introduced surface II, located in the supersonic region. If II is of the three-dimensional type at any given time, then it is unnecessary to specify any boundary conditions on this surface.

The mathematical formulation of the unsteady problem is obtained directly from the steady problem formulated above. One only has to write the total derivative in the differential equations with allowance for the explicit time dependence of functions. The introduction of the previously mentioned coordinate system (ξ, η, ϑ) will transform region I into one with constant boundaries, and the final equations acquire a derivative with respect to time:

$$\frac{\partial X}{\partial t} + A\frac{\partial X}{\partial \xi} + B\frac{\partial X}{\partial \eta} + C\frac{\partial X}{\partial \vartheta} + \Gamma = 0.$$

The mixed problem is stated for the region

$$t \geq t_0; \quad 0 \leq \xi \leq 1; \quad 0 \leq \eta \leq \eta_0; \quad 0 \leq \vartheta \leq 2\pi.$$

The initial conditions are specified for $t = t_0$ and the boundary conditions are given at $\xi = 0$ and $\xi = 1$.

It is assumed that surface $\eta = 0$ degenerates into the z axis. There is no need to specify boundary conditions at $\eta = 0$, but some measures must be taken in

*The latter is defined as that characteristic surface of all such surfaces having common points with the transition surface which is farthest removed from the latter.

order to remove the singularity introduced by the cylindrical coordinate system at r = 0. As we see, the problem reduces to a system with four independent variables. If the flow is axisymmetric, i.e., if there is no dependence on ϑ, then we again get a system with three variables. For simplicity this is the only case which is discussed subsequently. We note that if the bow part of the body is spherical, then for sufficiently large angles of attack axial symmetry may exist in region I relative to the direction of vector U_∞.

Thus, determination of the flow in regions I and II has been reduced to completely analogous problems for hyperbolic systems. Before these are solved by the difference method, we examine one more problem of the same type, namely, calculation of the flow past an arbitrary infinite cone. Such a flow is self-similar and all the functions retain constant values along rays drawn from the vertex. If in our substitution of variables we set $\zeta(\eta) = \eta$ and $\omega(\eta) = \pi/2$, then it takes the form

$$\begin{aligned} z &= \eta; & G &= G(\eta, \vartheta); \\ r &= G + \xi(F - G); & F &= F(\eta, \vartheta). \\ \varphi &= \vartheta; \end{aligned} \qquad /21$$

For a cone $G = \eta g(\vartheta)$ and $F = \eta f(\vartheta)$ should have a similar form. From this the direction of the ray drawn from the vertex is determined solely by the values of variables ξ and ϑ. This means that in the case of conical flow functions U, p and ρ will not depend on η and the term $\partial X/\partial\eta$ will drop out. Thus, in order to determine the flow about the cone one must solve the system of equations

$$A\frac{\partial X}{\partial \xi} + C\frac{\partial X}{\partial \vartheta} + \Gamma = 0$$

with boundary conditions at $\xi = 0$ and $\xi = 1$. This is very similar to the problem in region I and naturally suggests the use of the stabilization method for calculating the flow about a cone, in other words, carry out computations in region II for the complete system of equations until all the functions cease being functions of η. Experiment has shown that this is actually possible and, moreover, the method was found quite effective for calculation of conical flows [1].

3. Description of the Difference Scheme for Solving Two-Dimensional Problems of Flows Past Bodies

We shall now consider in detail the construction and investigation of the difference scheme using an unsteady axisymmetrical problem for illustration. The difference scheme for the other problems of this type formulated above is constructed similarly.

Firstly, we state the problem in more detail. We shall seek the solution of the system of quasilinear equations

$$\frac{\partial X}{\partial t} + A\frac{\partial X}{\partial \xi} + B\frac{\partial X}{\partial \eta} + \Gamma = 0,$$

where

$$X = \begin{Bmatrix} u \\ v \\ p \\ \rho \end{Bmatrix},$$

in region $t \geq t_0$; $0 \leq \xi = 1$; $0 \leq \eta \leq \eta_0$. The sought functions are the four-dimensional vector $X(t, \xi, \eta)$ and function $F(t, \eta)$, which determines the shape of the wave. The system's coefficients are A, B and Γ and the functions are ξ, η, X, F, F_η and F_t. We write the boundary conditions: 1) the condition at the body ($\xi = 0$) /22

$$U_n = 0 \quad \text{or} \quad n_z u + n_r v = 0,$$

where n_z and n_r are functions of η; 2) the condition at the wave ($\xi = 1$)

$$\begin{aligned}
&[U_\tau] = 0 && \text{or} && -\nu_r u + \nu_z v = U_{\tau\infty}; \\
&[p + \rho V_\nu^2] = 0 && \text{or} && p + \rho_\infty (U_{\nu\infty} - D)(\nu_z u + \nu_r v) = \\
& && && \quad p_\infty + \rho_\infty (U_{\nu\infty} - D) U_{\nu\infty}; \\
&[\rho V_\nu] = 0 && \text{or} && \rho (U_\nu - D) = \rho_\infty V_{\nu\infty}; \\
&\left[h + \frac{V_\nu^2}{2}\right] = 0 && \text{or} && h + \frac{U_\nu^2}{2} - U_\nu D = h_\infty + \frac{U_{\nu\infty}^2}{2} - U_{\nu\infty} D,
\end{aligned}$$

where $\nu = \{\nu_z, \nu_r\}$ is the normal to the shock wave and $\tau = \{-\nu_r, \nu_z\}$ is the tangent to it;

$$\begin{aligned}
&U_\tau = -\nu_r u + \nu_z v; && U_{\tau\infty} = -\nu_r u_\infty + \nu_z v_\infty; \\
&U_\nu = \nu_z u + \nu_r v; && U_{\nu\infty} = \nu_z u_\infty + \nu_r v_\infty; \\
&V_\nu = U_\nu - D; && V_{\nu\infty} = U_{\nu\infty} - D.
\end{aligned}$$

Components ν_z and ν_r of the normal are functions of η, F and F_η, but not of F_t. The wave velocity is a function of the above three variables and also of F_t.

Expressions for ν_z, ν_r and D are easily obtained from the shock wave equation in parametric form

$$z = \zeta(\eta) - F(t, \eta)\cos\omega(\eta); \quad r = F(t, \eta)\sin\omega(\eta);$$

$$\nu_z = -\frac{r_\eta}{\sqrt{z_\eta^2 + r_\eta^2}}; \quad \nu_r = \frac{z_\eta}{\sqrt{z_\eta^2 + r_\eta^2}},$$

where

$$z_\eta = \zeta_\eta + F\sin\omega\cdot\omega_\eta - F_\eta\cos\omega;\quad r_\eta = F\cos\omega\cdot\omega_\eta + F_\eta\sin\omega;$$

$$D = -\frac{z_t r_\eta - z_\eta r_t}{\sqrt{z_\eta^2 + r_\eta^2}} = v_z z_t + v_r r_t.$$

And since

$$z_t = -F_t\cos\omega;\quad r_t = F_t\sin\omega,$$

then

$$D = (-v_z\cos\omega + v_r\sin\omega)F_t.$$

We recall that D is the velocity with which the wave propagates normal to itself, while F_t is the velocity at which this wave moves along line η = const. /23

We now write the boundary conditions for $\eta = 0$ and $\eta = \eta_0$. For $\eta = 0$ we have the symmetry condition

$$\frac{\partial p}{\partial\eta} = \frac{\partial u}{\partial\eta} = \frac{\partial\rho}{\partial\eta} = 0;\quad v = 0.$$

There are no conditions at $\eta = \eta_0$ (this is a three-dimensional type line).

We now construct the difference scheme for this problem. We introduce in region $t \geq 0$; $0 \leq \xi \leq 1$; $0 \leq \eta \leq \eta_0$ a grid with lattice spacings $\Delta t = \tau$; $\Delta\xi = h_1 = 1/M$; $\Delta\eta = h_2 = \eta_0/L$. The difference scheme is written in two stages. First we assume that B = 0 and write the difference scheme for the equation

$$\frac{\partial X}{\partial t} + A\frac{\partial X}{\partial\xi} + \Gamma = 0.$$

We introduce the notation

$$f(n\tau, mh_1, lh_2) = f^n_{m,l};$$

$$x_1 = \frac{\tau}{h_1};\quad x_2 = \frac{\tau}{h_2}.$$

The difference scheme is here written for the four points marked by crosses in Fig. 4.

We approximate the derivatives:

$$\left(\frac{\partial X}{\partial t}\right)^{n+\frac{1}{2}}_{m+\frac{1}{2},\,l} = \frac{\left(X^{n+1}_{m+1} + X^{n+1}_{m}\right)_l - \left(X^{n}_{m+1} + X^{n}_{m}\right)_l}{2\tau};$$

$$\left(\frac{\partial X}{\partial \xi}\right)^{n+\frac{1}{2}}_{m+\frac{1}{2},\,l} = \frac{\alpha\left(X^{n+1}_{m+1} - X^{n+1}_{m}\right)_l + \beta\left(X^{n}_{m+1} - X^{n}_{m}\right)_l}{h_1},$$

where $\alpha + \beta = 1;\ \alpha, \beta > 0$. As is known, this scheme is stable at all the $\varkappa_1$.

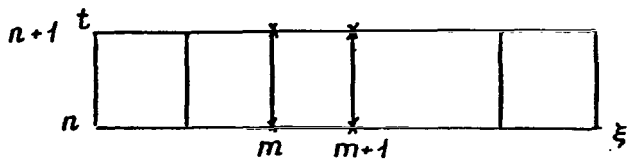

Figure 4.

Substituting these expressions into the differential equation and collecting like terms, the scheme takes the form

$$a^{n+\frac{1}{2}}_{m+\frac{1}{2}} X^{n+1}_{m+1,l} + b^{n+\frac{1}{2}}_{m+\frac{1}{2}} X^{n+1}_{m,l} = \pi^{n+\frac{1}{2}}_{m+\frac{1}{2},\,l}, \quad m = 0, 1, \ldots, M-1. \qquad /24$$

The functions X (on the upper layer) and F enter the right-hand side of π only. through the coefficients of matrix A and the free term Γ, and

$$a = E + 2\alpha\varkappa_1 A; \quad b = E - 2\alpha\varkappa_1 A.$$

We have obtained a quasilinear system of difference equations. These equations are supplemented inside the region by boundary conditions at the body and at the wave:

for $\xi = 0$,

$$n_{z,l}\, u^{n+1}_{0,l} + n_{r,l}\, v^{n+1}_{0,l} = 0;$$

for $\xi = 1$,

$$(v_r)^{n+1}_{M,l} u^{n+1}_{M,l} + (v_z)^{n+1}_{M,l} v^{n+1}_{M,l} = -(v_r)^{n+1}_{M,l} u_\infty + (v_z)^{n+1}_{M,l} v_\infty$$

etc.

The system of difference equations together with the boundary conditions yields a complete system for X^{n+1}_m. We shall solve it by the sweep method, assuming that the coefficients are known from the bottom layer.

We denote

$$\mu_m = \{\mu_{1,m}, \mu_{2,m}, \mu_{3,m}, \mu_{4,m}\}.$$

We consider an expression such as $\mu_m X_m = g_m$. The boundary condition at the body (for $\xi = 0$) can always be written as:

$$\mu_0 X_0 = g_0;$$
$$\mu_0 = \{n_z, n_r, 0, 0\}; \quad g_0 = 0.$$

We shall calculate μ_m and g_m in sequence, using equations for X^{n+1} (indices n + 1 and 1 are dropped),

$$\mu_m X_m = g_m;$$
$$a_{m+1/2} X_{m+1} + b_{m+1/2} X_m = \pi_{m+1/2}.$$

We assume for simplicity of mathematical manipulations that b is a nonsingular matrix. Then we find

$$\mu_m (b^{-1} a)_{m+1/2} X_{m+1} + g_m = \mu_m (b^{-1}_{m+1/2}) \pi_{m+1/2};$$
$$\mu^*_{m+1} = \mu_m (b^{-1} a)_{m+1/2};$$

$$g^*_{m+1} = \mu_m b^{-1}_{m+1/2} \pi_{m+1/2} - g_m. \qquad /25$$

These two vectors already are coefficients of the expression we need, but in order to make the sweep stable, they must be normalized. If, however, we calculate μ_{m+1} using the expression for μ^*_{m+1}, then the vector's components will increase without limit, since it can be shown that matrix b^{-1} a always has an eigenvalue whose absolute magnitude is greater than unity. On the other hand, it is easy to

see that the relationship $\mu^*_{m+1} X_{m+1} = g^*_{m+1}$ is equivalent to that which will be obtained by multiplying both sides of the equation by some arbitrary nonzero number. Hence it is expedient to normalize vector μ_{m+1} in such a manner that its norm will in some sense always be unity, i.e., to take for μ_{m+1} the expression

$$\mu_{m+1} = \frac{\mu^*_{m+1}}{\| \mu^*_{m+1} \|}.$$

Any norm can be used, but it is most convenient to use the maximum of the absolute value of the components of μ_m. For g_{m+1} we get

$$g_{m+1} = \frac{g^*_{m+1}}{\| \mu^*_{m+1} \|}.$$

In this case μ_{m+1} and g_{m+1} are bounded and the forward sweep is stable.

At the wave we get

$$\mu_{1,M} u_M + \mu_{2,M} v_M + \mu_{3,M} p_M = g_M,$$

since it can be shown that $\mu_{4,M}$ is always zero. Together with the boundary conditions this yields a system of equations for X_M and D which can be easily solved by iteration. First, specifying D, we find u, v and p from the system of equations (subscript M has been dropped):

$$\mu_1 u + \mu_2 v + \mu_3 p = g; \quad -\nu_r u + \nu_z v = U_{\tau\infty};$$
$$\rho_\infty V_{\nu\infty} (\nu_z u + \nu_r v) + p = \rho_\infty V_{\nu\infty} U_{\nu\infty} + p_\infty.$$

Whereupon we find ρ and again D:

$$\rho = \rho_\infty \frac{U_{\nu\infty} - D}{U_\nu - D}; \quad D = \frac{h - h_\infty + \frac{U_\nu^2 - U_{\nu\infty}^2}{2}}{U_\nu - U_{\nu\infty}}.$$

Upon termination of iterations at the wave we have found X and D. Using D we find F_t, after which it is possible to find F^{n+1} by integrating equation $\frac{\partial F}{\partial t} = F_t$; $X_{M-1}, X_{M-2}, \ldots, X_0$ is found by a reverse sweep. We thus obtain a

/26

solution at the upper layer on the assumption that the coefficients were determined for the bottom layer.

In order to be able to implement this scheme of solution, it is necessary that the sweep be stable. Stability of the sweep in this case is identical with the requirement that the system be well conditioned for X_m. It can be shown that satisfactory conditionality is determined by the signs of the eigenvalues of matrix A, namely, that among the eigenvalues of this matrix there should be three negative and one positive value. This is due to the direction of the characteristic cone and geometrically can be interpreted as follows: one in the firection of increasing ξ, and three in the direction of decreasing ξ. Since the central characteristic (the particle path) is a multiple, actually two characteristics are directed toward the body and one toward the wave. This direction of the characteristics is in agreement with the number of boundary conditions at the wave and the body. In fact, at the body there is a single relationship for X, and one characteristic departs from it. At the wave there actually exist three relationships for X and three characteristics depart from the wave.

The difference scheme was written on two assumptions: 1) that the coefficients of the equations and of the boundary conditions do not depend on X^{n+1} and F^{n+1} and 2) that B = 0. The first assumption is removed by repeating the solution of the system, taking now the coefficients from the first iteration. This can also be done for B. In order to better understand how this is done, we again consider the particular case of A = 0 and Γ = 0:

$$\frac{\partial X}{\partial t} + B\frac{\partial X}{\partial \eta} = 0.$$

We first write the implicit system, expressing the derivatives as (Fig. 5):

$$\left(\frac{\partial X}{\partial t}\right)_l^{n+\frac{1}{2}} = \frac{X_l^{n+1} - X_l^n}{\tau};$$

$$\left(\frac{\partial X}{\partial \eta}\right)_l^{n+\frac{1}{2}} = \frac{\alpha(X_{l+1}^{n+1} - X_{l-1}^{n+1}) + \beta(X_{l+1}^n - X_{l-1}^n)}{2h_2}.$$

Substituting them into the differential equations we get /27

$$X_l^{n+1} = X_l^n - \frac{\varkappa_2 B}{2}[\alpha(X_{l+1} - X_{l-1})^{n+1} + \beta(X_{l+1} - X_{l-1})^n].$$

We investigate the above scheme for stability by the Fourier method on the assumption that all the eigenvalues σ_k of matrix B are real (k = 1, 2, 3, 4).

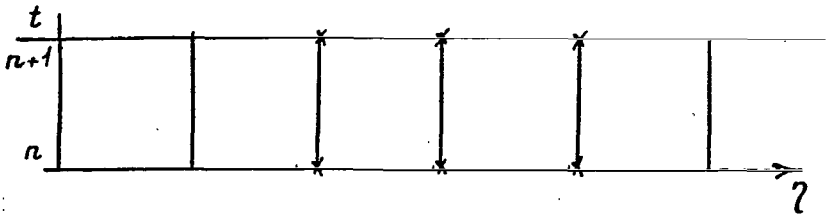

Figure 5.

Setting, as usual, $X_l^n = e^{il\psi} X_0^n$, $X_l^{n+1} = \lambda e^{il\psi} X_0^n$, we find an expression for λ. Eigenvalue λ_k corresponding to the value of σ_k is written as:

$$\lambda_k = \frac{1 - i\beta\varkappa_2 \sigma_k \sin\psi}{1 + i\alpha\varkappa_2 \sigma_k \sin\psi},$$

whence it is seen that $|\lambda_k| \leq 1$ if $\alpha \geq \beta$, i.e., the scheme will always be stable for $\alpha \geq \beta$. For $\alpha = \beta$ it will be of the second order of accuracy. If α is slightly greater than β, then it has a second-order residual term with the coefficient $(\alpha - \beta)$. All this pertains to a scheme regarded as implicit over the entire layer. We shall now solve it by iterations and calculate the value of X_l^{n+1} from the above formula, taking first the terms X_{l+1}^{n+1} in the righthand side at the bottom layer n; for the next iteration we shall use their values from the preceding iteration. In order to formalize this, we introduce the following notation: let q be the number of the iteration, and let $X^{n+(q)}$ be the value at the (n + 1)th layer at the qth iteration; here, if a total of Q iterations is made, then $X^{n+(Q)}$ is regarded as equal to the value of X at the (n + 1)th layer $X^{n+(Q)} = X^{n+1}$. Then our formula takes the form

$$X^{n+(q+1)} = X^n - \frac{\varkappa_2 B}{2}\{\alpha(X_{l+1} - X_{l-1})^{n+(q)} + \beta(X_{l+1} - X_l)^n\},$$

it being assumed here that $X^{n+(0)}$ is simply X^n, i.e., that the zeroth iteration is the preceding layer.

We now integrate using the formula. We first assume that q = 0, calculate $X^{n+(1)}$, then take q = 1, etc. up to q = Q. /28

It can be shown that this scheme will still be of the second order of accuracy for $\alpha = \beta$ and will be close to it for an α slightly greater than β if $q \geq 2$. This means that, starting with the second iteration, the second order of accuracy is retained, hence not more than two iterations are needed from the point of view of accuracy.

We now consider the question of stability. When the equations contained simply X^{n+1}, i.e., when the scheme was implicit, it was stable for any ratio of lattice spacings when $\alpha \geq \beta$. When, however, the system at the upper layer is solved by iterations and the computations are stopped at some given iteration, then we in substance have an entirely different difference scheme, and its stability must be examined anew.

We again denote $X_l^{n+(q)} = \lambda^{(q)} e^{il\psi} X_0^n$ and substitute it into the scheme. Then upon division by $e^{il\psi}$, we get

$$\lambda^{(q+1)} X_0^n = X_0^n - \varkappa_2 B\{(\alpha\lambda^{(q)} + \beta) i \sin\psi\} X_0^n .$$

Since vector X_0^n is different from zero, we can write an equation which $\lambda^{(q+1)}$ and $\lambda^{(q)}$ will satisfy,

$$\mathrm{Det}\{(\lambda^{(q+1)} - 1)E + i\varkappa_2(\alpha\lambda^{(q)} + \beta)\sin\psi B\} = 0.$$

If it is remembered that matrix B has eigenvalues σ_k, then the expression for $\lambda^{(q+1)}$ can be found in terms of the preceding in explicit form

$$\lambda^{(q+1)} = 1 - i\varkappa_2\sigma_k(\alpha\lambda^{(q)} + \beta)\sin\psi .$$

Now this expression is used for calculating λ corresponding to the Qth iteration and then the stability condition is checked.

Setting $\lambda^{(0)} = 1$, we shall find $\lambda^{(1)}$ and $\lambda^{(2)}$ and clarify under which conditions the absolute value of λ will be smaller than unity. We find

$$\begin{aligned}
\lambda^{(1)} &= 1 - i\varkappa_2\sigma_k\sin\psi ;\\
|\lambda^{(1)}|^2 &= 1 + \varkappa_2^2\sigma_k^2\sin^2\psi \geq 1;\\
\lambda^{(2)} &= 1 - \alpha\varkappa_2\sigma_k^2\sin^2\psi - i\varkappa_2\sigma_k\sin\psi;\\
1 - |\lambda^{(2)}|^2 &= [(2\alpha - 1) - \alpha^2(\varkappa_2\sigma_k\sin\psi)^2](\varkappa_2\sigma_k\sin\psi)^2 .
\end{aligned}$$

The value of $|\lambda^{(2)}|$ will be smaller than unity if

$$2\alpha - 1 - \alpha^2(\varkappa_2\sigma_k\sin\psi)^2 \geq 0.$$

If $\alpha > \frac{1}{2}$, then $|\varkappa_2 \sigma_k \sin\psi| \leq \frac{\sqrt{2\alpha-1}}{\alpha}$ or $\varkappa_2|\sigma_k| \leq \frac{2\alpha-1}{\alpha}$, k = 1, 2, 3, 4. Since the σ_k may be different, the stability condition is taken as /29

$$\varkappa_2 = \frac{\tau}{h_2} \leq \frac{\sqrt{2\alpha-1}}{\alpha} \cdot \frac{1}{\max\limits_k |\sigma_k|}.$$

Without the first multiplier, this would have been Courant's condition for the explicit scheme. The presence of the multiplier $\sqrt{2\alpha-1}/\alpha$ decreases the permissible lattice spacing. As a result, we obtain a condition for the stability of the difference system at A = 0.

Now to construct the complete difference scheme we combine the two above schemes, writing an implicit sweep scheme in the ξ direction and an iteration scheme in the η direction. First we introduce the shift operators:

$$SX^{n+(q)}_{m,l} = X^{n+(q)}_{m+1,l}; \quad TX^{n+(q)}_{m,l} = X^{n+(q)}_{m,l+1}; \quad IX^{n+(q)}_{m,l} = X^{n+(q)}_{m,l}.$$

Then the difference scheme can be written as:

$$(S+I)(X^{n+(q+1)}_{m,l} - X^{n}_{m,l}) + 2\varkappa_1 A^{n+\frac{q}{2}}_{m+\frac{1}{2},l}(S-I)(\alpha X^{n+(q+1)}_{m,l} + \beta X^{n}_{m,l}) +$$

$$+ \frac{\varkappa_2}{2} B^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l}(S+I)(T-T^{-1})(\alpha X^{n+(q)}_{m,l} + \beta X^{n}_{m,l}) + 2\tau\Gamma^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l} = 0,$$

where

$$A^{n+\frac{q}{2}}_{m+\frac{1}{2},l} = A\left(\frac{X^{n+(q)}_{m+1,l} + X^{n+(q)}_{m,l} + X^{n}_{m+1,l} + X^{n}_{m,l}}{4}, \ldots\right).$$

The same scheme, in the form transformed for solution:

$$\{(E+2\alpha\varkappa_1 A)S + (E-2\alpha\varkappa_1 A)I\}^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l} X^{n+(q+1)}_{m,l} =$$

$$\{S+I-2\beta\varkappa_1 A(S-I)\}^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l} X^{n}_{m,l} - \frac{\varkappa_2}{2} B^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l}(S+I)(I-T^{-1}) \times$$

$$(\alpha X^{n+(q)}_{m,l} + \beta X^{n}_{m,l}) - 2\tau\Gamma^{n+\left(\frac{q}{2}\right)}_{m+\frac{1}{2},l}.$$

We first set q = 0, substitute into the coefficients the values of functions from the bottom layer and calculate the first iteration, i.e., for all the l we find $X_{m,l}^{n+(1)}$, $F_l^{n+(1)}$, after which we substitute into the coefficients the value of the first iteration and repeat the sweep again. /31

We shall examine the stability of the difference scheme. We set

$$X_{m,l}^{n+(q)} = \lambda^{(q)} e^{i(m\varphi + l\psi)} X_{0,0}^{n}.$$

Investigation of its stability for the case of noncommutating A and B is quite difficult. If A and B commutate (which is not true of gasdynamics), then examination of the stability shows that the absolute eigenvalue $\lambda^{(q)}(\varphi, \psi)$ can be greater than unity in a very narrow range of frequencies (φ, ψ). To ensure total stability, one must introduce into the scheme a second difference $X_{m,l}^{n}$ with respect to l, with a small coefficient. /32

Figures 6 and 7 depict some results obtained using the above scheme. Figure 6 shows the shapes of shock waves and the sonic curves with the characteristics adjoining to them for the flow past a sphere at Mach numbers M_∞ from 1.25 to 4. One's attention is attracted by the change in the location of characteristics on passing from $M_\infty = 2$ to $M_\infty = 4$. Figure 7 shows the shape of the wave and of the sonic line for the flow past a body with a negative curvature segment at $M_\infty = 10$. It is interesting to note that the location of the characteristics here is the same as for a sphere at markedly lower M_∞.

4. Calculation of Flow in the Supersonic Region

In the supersonic region one usually deals with an equation with three variables

$$A\frac{\partial X}{\partial \xi} + B\frac{\partial X}{\partial \eta} + C\frac{\partial X}{\partial \vartheta} + \Gamma = 0$$

and the solution of the problem which was stated for it is carried out using a scheme similar to that described above. The basic difference consists in the fact that vector X here has five components and the solution of equations at the wave is carried out somewhat differently from the technical point of view. As was noted previously, the qualitative features of the flow may require in some cases changes and refinements of the difference scheme. Thus, for example, in axisymmetric flow past a blunt cone or cylinder the entropy at the body retains the value corresponding to a plane shock. If we follow the streamlines we will find that each of them comes increasingly closer to the body's surface. However, the entropy at each streamline is constant and depends on the inclination of the shock wave at

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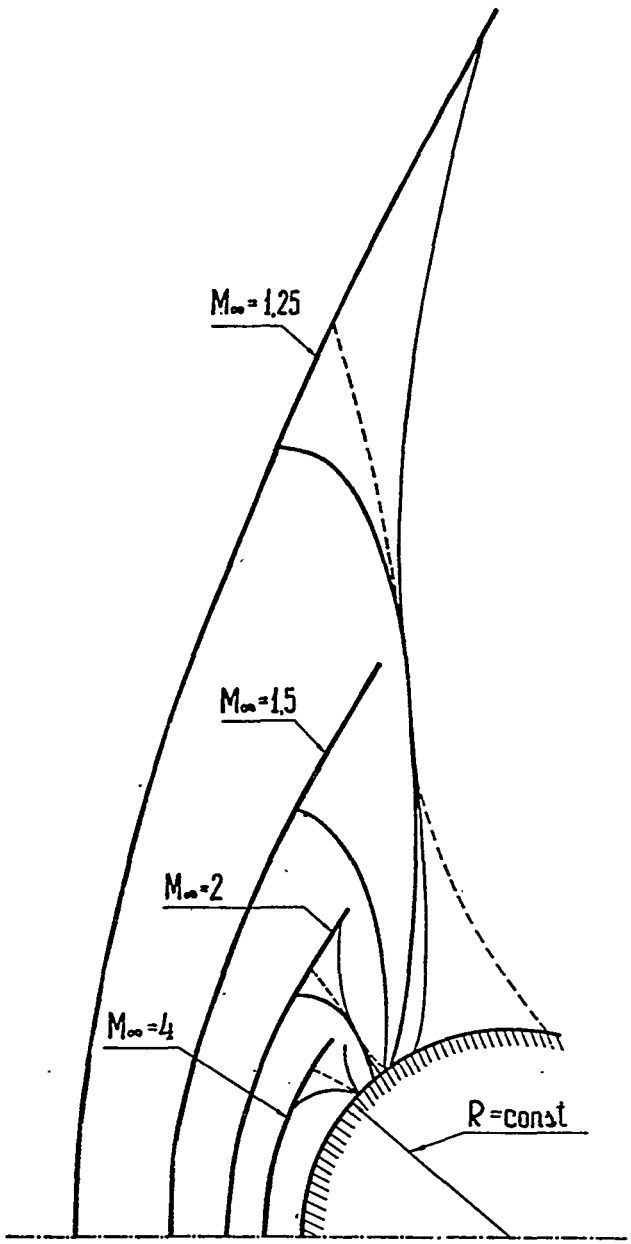

Figure 6.

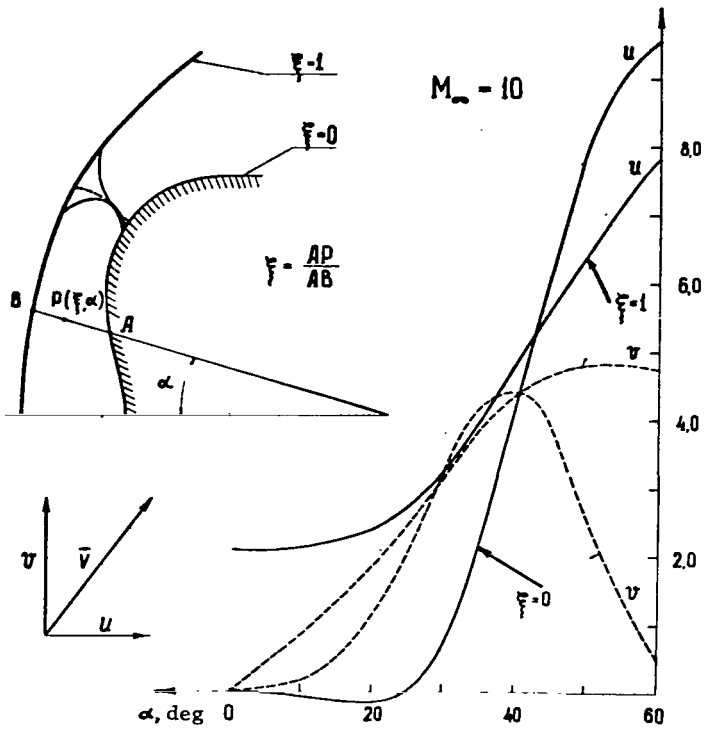

Figure 7.

the point where the latter is intersected by the streamline. Hence the distribution of the entropy along the normal to the body in the vicinity of the latter repeats its distribution along the wave, but in a very narrow zone, which results in the appearance of sharp entropy gradients, which increase continuously as one moves downstream. To calculate this so-called entropy layer one must take special measures, for example, make the grid near the body more dense.

Figures 8 and 9 show the distributions of pressure and density as functions of ξ in different sections z = const for a cone with a spherical nose with an apex half-angle of 15° at $M_\infty = 10$. It is seen from Fig. 8 that for $z \geq 25$ the pressure distribution is virtually independent of z and coincides with the pressure on the unblunted cone. At the same time the density distribution (Fig. 9) follows the entropy variation. The density maximum (and accordingly the entropy minimum) is attributable to the nonmonotonicity of the inclination of the shock wave relative to z, arising due to over-expansion of the flow in the region where the sphere joins the cone. /33

Depending on the shape of the body and of its nose part, there may be several density maxima and minima, or there may be none, as in the flow past a cylinder with a spherically blunted nose at $M_\infty = 10$ (Figs. 10 and 11). Figure 11 depicts the density distribution about the surface of the cylinder as a function of r - 1 (the

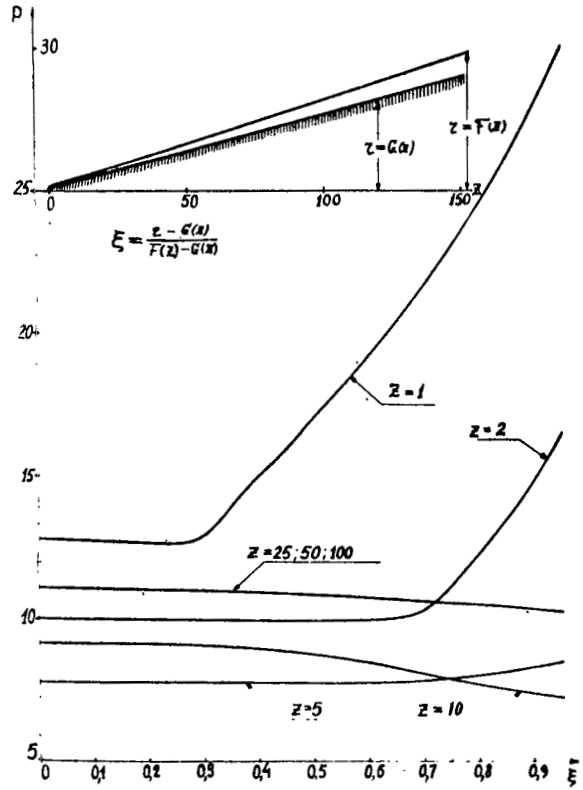

Figure 8.

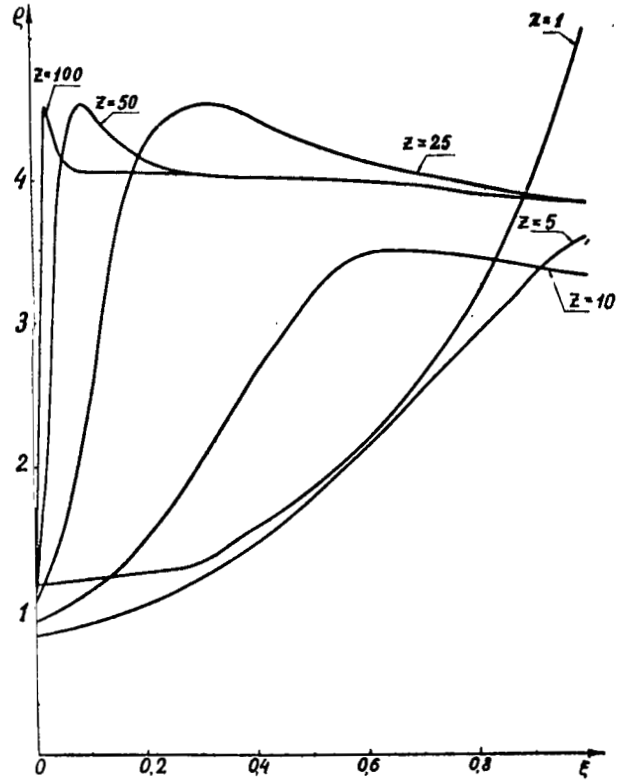

Figure 9.

*Translator's Note: Commas represent decimal points.

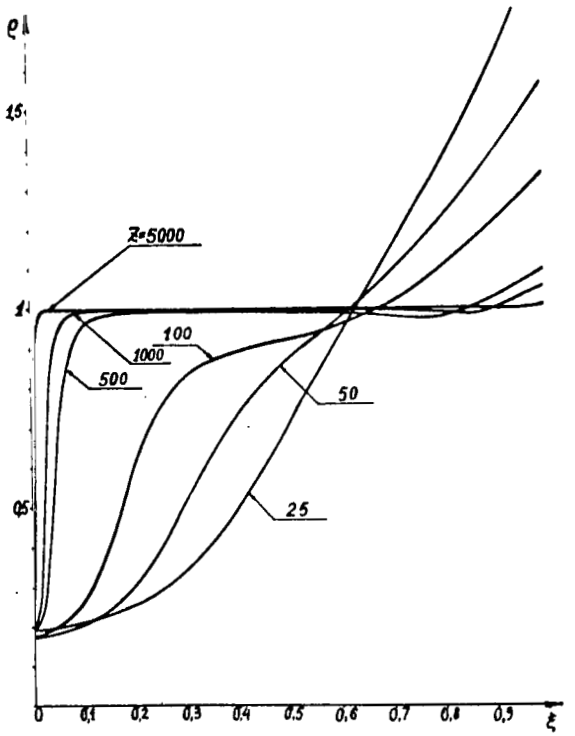

Figure 10.

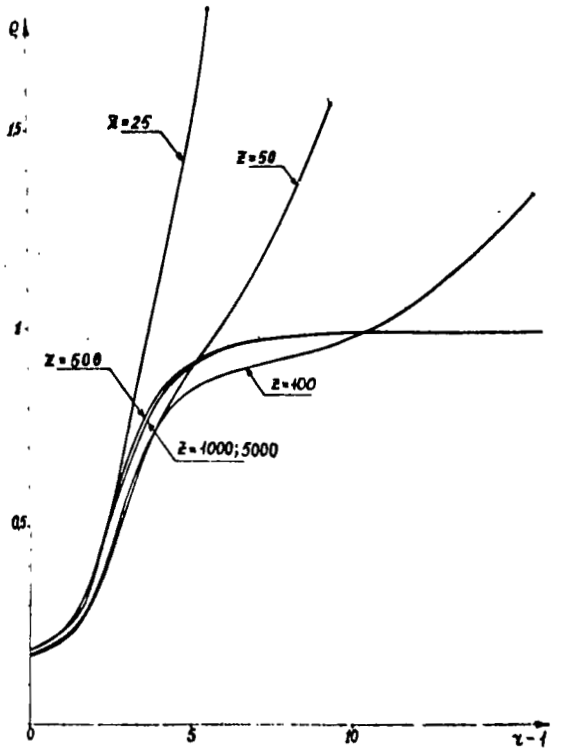

Figure 11.

cylinder radius is unity). Here one can clearly see the formation of an asymptotic density distribution for large z.

In the case of three-dimensional flow the pattern becomes even more complicated by the difference in behavior of the function relative to ξ for different ϑ. Some results of calculations for a blunted cone at $M_\infty = 4$ with an apex half-angle 10° at angles of attack of 5, 10 and 15° are presented in Figs. 12-15. In particular, one's attention is attracted by the sharp difference in behavior of the density on the windward and leeward side as early as at a relatively small distance from the nose (see Fig. 14).

Let us consider several more features peculiar to flow past a cone. The corresponding system of equations has the form

$$A \frac{\partial X}{\partial \xi} + C \frac{\partial X}{\partial \vartheta} + \Gamma = 0.$$

Investigation shows that it is elliptical at points where the velocity component normal to the ray $\xi = \text{const}$, $\vartheta = \text{const}$ has an absolute magnitude smaller than the speed of sound, and it is hyperbolic when the opposite is true. Further, for a circular cone at small angles of attack the equations are always elliptical in the region between the wave and the body, but as the angle of attack is increased regions of hyperbolicity arise near the wave, and these are due primarily to a sharp rise in the velocity component w. As was pointed out previously, it is most convenient to calculate the flow past a conical body by the stabilization method, for which the existence or nonexistence of regions of hyperbolicity is inconsequential. However, one must take into account some other features of the flow, for example, the behavior of the constant-entropy line. As has already /39
been shown by Ferri, the function $S(\xi, \vartheta)$ in the flow past a circular cone has a singularity inside the region $0 \leq \xi \leq 1$, $0 \leq \vartheta \leq 2\pi$. It can be shown in addition that at the cone surface $\frac{\partial S}{\partial \xi} = \infty$ and a sharp change in the flow variables occurs near the surface. This layer with large gradients is usually called the vortex layer. The existence of singular entropy points and the behavior of the function near the surface must be taken into account in constructing the computational algorithm. When this is done it is possible to obtain satisfactory results when calculating flows with singularities and to also detect a number of fine details. Thus, Figs. 16 and 17 depict the shapes of shock waves and of constant-entropy curves for two cases of flow around a circular cone. Figure 16 presents results for the flow past a cone with apex half-angle $\beta = 25^\circ$ at angle of attack $\alpha = 20^\circ$ and $M_\infty = 3$. Figure 17 is plotted for $\beta = 35^\circ$, $\alpha = 10^\circ$ and $M_\infty = 5$.

It is emphasized that all the above results pertaining to singularities and flow details were obtained by a pure difference method, without using any asymptotic expansions or other analytical methods to account for the singularities.

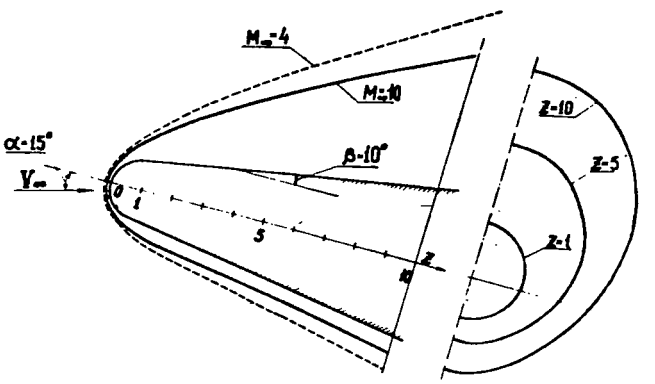

Figure 12.

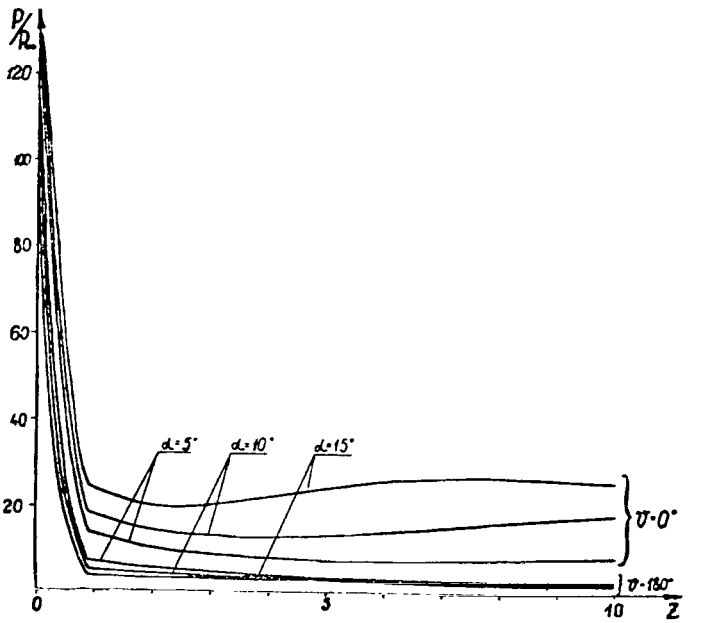

Figure 13.

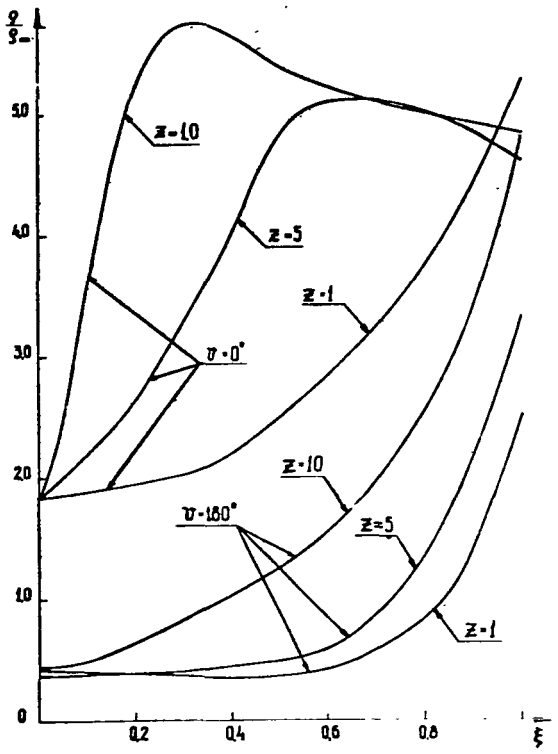

Figure 14.

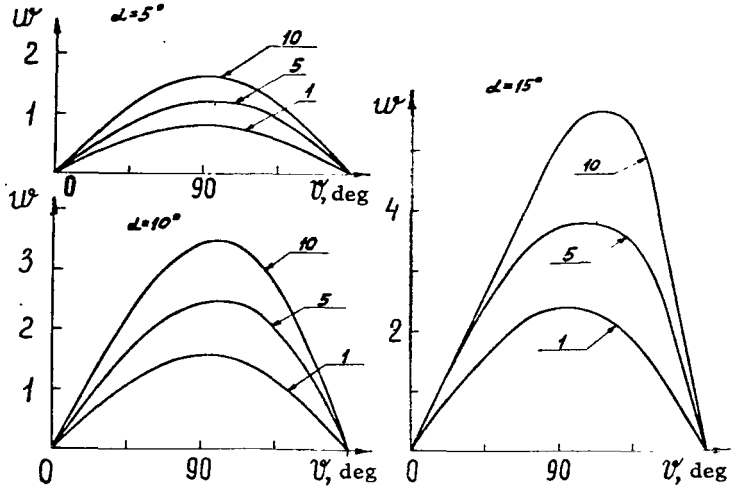

Figure 15.

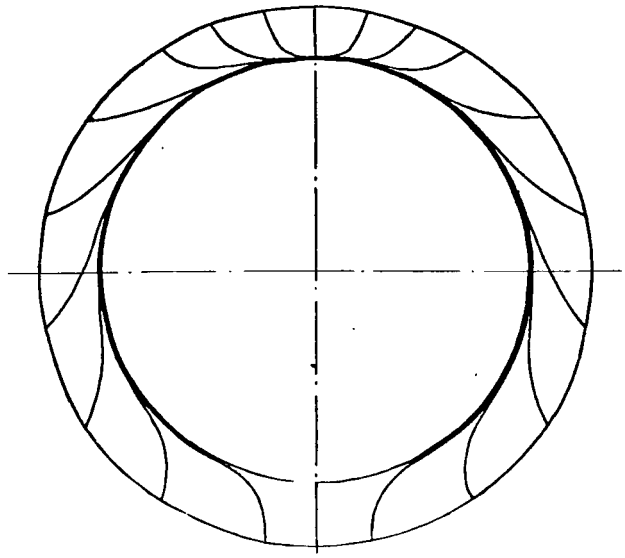

Figure 16.

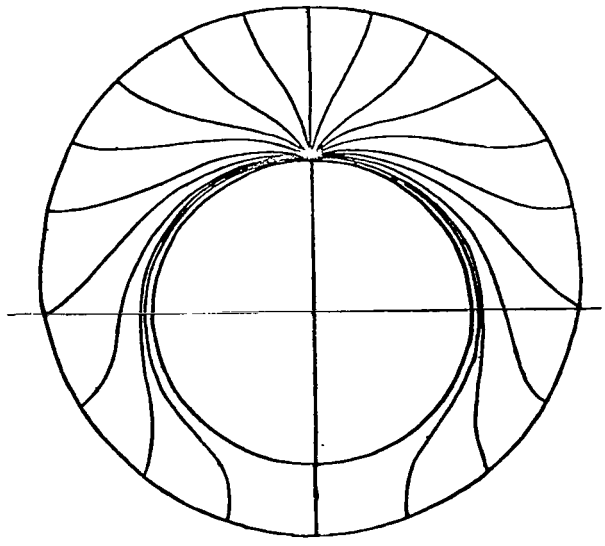

Figure 17.

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METHODS OF SOLVING UNSTEADY PROBLEMS OF GASDYNAMICS

B.F. D'yachenko

1. Gasdynamic Equations

The state of a continuous medium, be it a liquid or gas, is described by the distribution in space of various physical variables — the velocity u, density ρ and internal energy per unit mass ε. Usually the pressure p, entropy S, temperature T, etc. are also used, but all of them are functions of ρ and ε. The change in the state of a continuous medium with time t is described by the known gasdynamic equations which, for the case when all the functions depend solely on the space coordinate x, can be written in the form [1] /40

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0;\quad \frac{\partial \rho u}{\partial t}+\frac{\partial (p+\rho u^2)}{\partial x}=0;\\ &\frac{\partial}{\partial t}\rho\left(\varepsilon+\frac{u^2}{2}\right)+\frac{\partial}{\partial x}u\left(p+\rho\left(\varepsilon+\frac{u^2}{2}\right)\right)=0, \end{aligned}\right\} \qquad (1.1)$$

where $p = p(\rho, \varepsilon)$ is specified. It is useful to clarify the physical meaning of the system of equations in (1.1). Each equation of this system has the form

$$\frac{\partial f}{\partial t}+\frac{\partial g}{\partial x}=0, \qquad (1.2)$$

where f is the distribution density of mass, momentum or energy, respectively, while g is the flux density of these same quantities. System of equations (1.1) is nothing other than the mathematical expression of the laws of conservation of mass, momentum and energy. In fact, let us consider in the plane x, t an infinitesimal rectangle with sides Δx and Δt (Fig. 1). For definiteness, we shall consider the mass. Then $f\Delta x$ is the mass contained within the interval Δx, while $f_2\Delta x - f_1\Delta x = \Delta_t f\Delta x$ is the change in this mass during time Δt. On the other hand, /41 $g_1\Delta t$ is the quantity of mass arriving into the interval during this time, and $g_2\Delta t$ is the quantity of mass departing from it during the same time, i.e., the change in mass of the interval under study is

$$g_1\Delta t - g_2\Delta t = -\Delta_x g\Delta t.$$

By virtue of the law of conservation the two expressions for the change of mass should be equal to one another

$$\Delta_t f\Delta x = -\Delta_x g\Delta t.$$

Dividing the above equality by $\Delta x \Delta t$ and letting Δx and Δt go to zero, we get Eq. (1.2) i.e., system of equations (1.1).

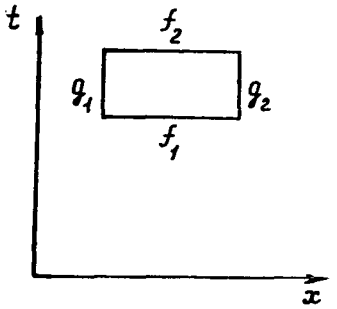

Figure 1.

We shall also use other forms of the system of equations of gasdynamics. Using obvious transformations, system (1.1) can be reduced to the form

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0;\\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0;\\ &\frac{\partial \varepsilon}{\partial t} + u\frac{\partial \varepsilon}{\partial x} + \frac{p}{\rho}\frac{\partial u}{\partial x} = 0. \end{aligned}\right\} \qquad (1.3)$$

The derivative $\frac{\partial p}{\partial x}$ in the second of these equations can be written as

$$\frac{\partial p}{\partial x} = p_\rho \frac{\partial \rho}{\partial x} + p_\varepsilon \frac{\partial \varepsilon}{\partial x},$$

where p_ρ and p_ε are partial derivatives of the function $p(\rho, \varepsilon)$, and then the system is linear relative to the derivatives of the principal functions ρ, u and ε.

The system of equations in (1.3) is called the Eulerian description. Here x corresponds to a given point in space. One may transform the expressions to the so-called Lagrangian description, replacing x by the coordinate s corresponding to a given mass, a gas "particle." It is simplest to do this as follows [2]. The first of equations (1.1) is the condition for the expression $\rho dx - \rho u dt$ to be a total differential. Hence it is possible to convert from x, t to the new coordinates s, t' defined as

$$ds = \rho dx - \rho u dt; \quad dt' = dt,$$

whence

$$\frac{1}{\rho}\frac{\partial}{\partial x} = \frac{\partial}{\partial s}; \quad \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{\partial}{\partial t'},$$

and, dropping the prime of t, we rewrite Eq. (1.3) in the form which we need

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho^2\frac{\partial u}{\partial s} = 0;\\ &\frac{\partial u}{\partial t} + \frac{\partial p}{\partial s} = 0;\\ &\frac{\partial \varepsilon}{\partial t} + p\frac{\partial u}{\partial s} = 0. \end{aligned}\right\} \qquad (1.4)$$

The fact that actually represents the mass is seen form the expression

$$s = s_0 + \int_{x_0}^{x} \rho \, dx,$$

which is valid along t = const. The Eulerian coordinate x of the particle now becomes the unknown function, defined as

$$\frac{\partial x}{\partial t} = u.$$

Finally, we derive still another form of the system of gasdynamic equations — the characteristic form. We multiply the first of equations (1.3) by p/ρ^2 and subtract it from the third, yielding

$$\frac{\partial \varepsilon}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} + u\left(\frac{\partial \varepsilon}{\partial x} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial x}\right) = 0.$$

Equation $d\varepsilon - \frac{p(\rho, \varepsilon)}{\rho^2} d\rho = 0$ has a solution $S(\rho, \varepsilon) = \text{const}$, which is called the entropy. Hence the above equation can be written as

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0. \tag{1.5}$$

Taking further the linear combination of the three equations (1.3) with coefficients p_ρ, $\pm \rho c$ and p_ε, where

$$c = \sqrt{p_\rho + \frac{p p_\varepsilon}{\rho^2}},$$

we get

$$\frac{\partial p}{\partial t} + (u \pm c)\frac{\partial p}{\partial x} \pm \rho c\left(\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right) = 0. \tag{1.6}$$

Equations (1.5) and (1.6) have the following remarkable property. The /43
lefthand side of Eq. (1.5) has the derivative dS/dt, taken in the direction

$$\frac{dx}{dt} = u. \tag{1.7}$$

It can hence be stated that along line (1.7), which is called a characteristic, we should have

$$dS = 0, \tag{1.8}$$

i.e., the entropy is conserved. The lefthand side of Eq. (1.6) contains derivatives dp/dt and du/dt, taken in the direction

$$\frac{dx}{dt} = u \pm c. \tag{1.9}$$

Hence the relationships

$$dp \pm \rho c du = 0. \tag{1.10}$$

must be satisfied along the characteristics (1.9).

For the system of gasdynamic equations one states the Cauchy problem, i.e., the solution is known at initial time t = 0 and it is required to find the solution for $t > 0$. Below we shall consider some approximate numerical methods for solving this problem.

2. The Method of Characteristics

The method is based on the property of hyperbolicity of the three equations of gasdynamics, which consists in the fact that it has three families of real characteristics (1.7) and (1.9). We describe this method first in the simplest form, when everywhere $S(\rho, \varepsilon) = S_0 = \text{const}$. In this case Eq. (1.8) is satisfied automatically and we need use only Eqs. (1.9) and (1.10). Let the solution be known at some collection of points (in particular, initial conditions).

Let us consider some point M_1 (Fig. 2) and draw through it a ray (characteristic (1.9)) in the direction of increasing t:

$$\frac{x - x_1}{t - t_1} = u_1 + c_1.$$

The subscript denotes the point where the corresponding quantity is taken. Let point M_2 be the closest to point M_1 from the right. We draw through point M_2 /44 the characteristic of the second of the families in (1.9)

$$\frac{x - x_2}{t - t_2} = u_2 - c_2.$$

The point of intersection of both rays is denoted by M*. Equations (1.10) should be satisfied along the characteristics. Replacing the differentials therein by finite increments, we get

$$(p^* - p_1) + \rho_1 c_1 (u^* - u_1) = 0;$$
$$(p^* - p_2) + \rho_2 c_2 (u^* - u_2) = 0,$$

i.e., a system of algebraic equations for p* and u*. Finally, ρ^* and ε^* are found from

$$p(\rho^*, \varepsilon^*) = p^*; \quad S(\rho^*, \varepsilon^*) = S_0.$$

Thus, knowing the solution at the points M_1 and M_2, we obtain it at the point M* corresponding to a higher value of t. And this is so for each pair of points. Then we get the points M* thus obtained as reference points and make another time step, etc., up to the necessary times t.

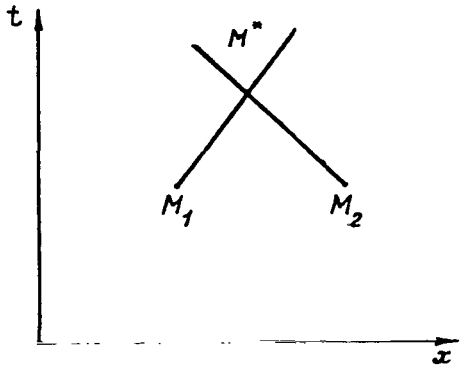

Figure 2.

Obviously, we get an approximate solution whose accuracy increases with the density of the points. The above method is of the first order of accuracy, i.e., the error is proportional to the first power of the distance between pairs M_1 and M_2 and corresponds to the Euler's method of broken lines for solving ordinary differential equations. As for the latter, the accuracy can be improved by recalculation, i.e., by repeating each cycle of the calculations, taking for the slope of the characteristics $u \pm c$ and coefficients ρc half the sum of their values in M* and the initial values. In addition, the values obtained by this recalculation can be used to /45

judge to some extent the accuracy of the initial calculation and to take appropriate measures by adding or dropping computational points.

In order to better explain the substance of the method of characteristics, we considered the case of $S(\rho, \varepsilon) \equiv \text{const}$. The general case requires some additional computations. Namely, after the coordinates of M* (Fig. 2) and u* are found, one should draw through point M* the line

$$\frac{x - x^*}{t - t^*} = u^*,$$

corresponding to Eq. (1.7) up to intersection with segment M_1M_2 at point M_3 (Fig. 3).

Values of S at points M_1 and M_2 are used for interpolation to point M_3 and it is assumed by virtue of Eq. (1.8) that

$$S^* = S_3.$$

Then ρ^* and ε^* we found from

$$p(\rho^*, \varepsilon^*) = p^*; \quad S(\rho^*, \varepsilon^*) = S^*.$$

As before, the results can be improved by recalculation.

3. Discontinuous Solutions

The above computational processes cannot be always extended over a sufficient time. This is not due to shortcomings of the method of characteristics, but due to nonlinearity of gasdynamic equations, as a result of which the characteristics of one family touch or intersect at some time instant [2]. The manner in which such a singularity can occur in the solution is shown using the simple quasilinear equation

$$\frac{\partial u}{\partial t} - u \frac{\partial u}{\partial x} = 0.$$

This equation is solved simply. Along characteristic

$$\frac{dx}{dt} = -u$$

one must satisfy /46

$$\frac{du}{dt} = 0,$$

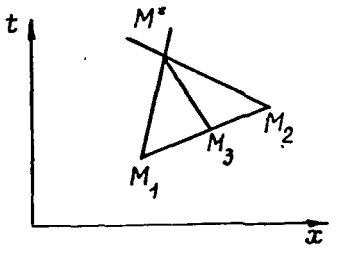

Figure 3.

i.e., $u = u_0 = \text{const}$ along straight line $x = x_0 - u_0 t$. Let the initial conditions be such that at some segment $u(0, x) = x$. It is easy to see that all the characteristics originating at points of this segment intersect at point $x = 0$, $t = 1$. Each characteristic contributes its value and the solution is no longer unique; an arbitrary discontinuity occurs, since the initial conditions outside of the above segment are arbitrary.

Precisely the same effect can be observed in solutions of the system of gasdynamic equations. But the latter describes an actual physical process which cannot somehow cease at a certain instant. The way out of this situation is shown by physical experiments. It is necessary to introduce discontinuous solutions, i.e., one must assume that values of ρ, u and ε may be discontinuous at some curves $x = x(t)$. Obviously, the differential equations are meaningless at these curves and should be replaced by other relationships, reflecting the same laws of conservation of mass, momentum and energy.

As in Sec. 1, we consider an infinitesimal rectangle with sides Δx and Δt. Let the discontinuity line intersect this rectangle diagonally (Fig. 4). Denoting again the distribution density of some quantity by f and the flux density by g and repeating the considerations of Sec. 1, we get

$$(f_l - f_r)\Delta x = (g_l - g_r)\Delta t.$$

We divide the above expression by Δt and let the latter approach zero. We get

$$(f_l - f_r)D = (g_l - g_r),$$

where

$$D = \frac{dx}{dt}$$

is the rate of propagation of the discontinuity.

Replacing f and g by their expressions and denoting the differences of the corresponding quantities for both sides of the discontinuity by brackets, we write the needed conditions on the discontinuity

$$[\rho]D = [\rho u]; \qquad (3.1)$$

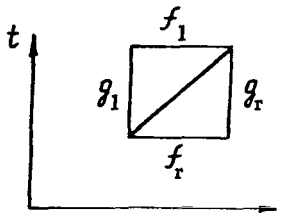

Figure 4.

$$\left.\begin{aligned} [\rho u]D &= [p + \rho u^2]; \\ \left[\rho\left(\varepsilon + \frac{u^2}{2}\right)\right]D &= \left[u\left(p + \rho\left(\varepsilon + \frac{u^2}{2}\right)\right)\right]. \end{aligned}\right\} \quad (3.1)$$ /47

Let us now see how the method of characteristics changes in the presence of discontinuous solutions. In addition to the three families of characteristics with relationships holding on them, we now have discontinuity lines along which Eq. (3.1) should be satisfied. The latter contain seven quantities: ρ, u, ε to the left and ρ, u, ε to the right of the discontinuity, and D, the rate of propagation of the discontinuity. For determining these unknowns we have the three equations (3.1). The four additional relations needed should be supplied by characteristics "arriving" at the discontinuity line. Hence a discontinuous solution satisfying Eqs. (3.1) can exist only in the case when precisely four characteristics arrive at the discontinuity line. This condition is equivalent to the known thermodynamic condition that the entropy cannot decrease and it will subsequently be regarded as satisfied.

There are two types of lines of discontinuity — shock waves and contact discontinuities. The former are typified by the fact that of the four characteristics arriving at the shock three do so from one side and the fourth from the other. In the latter case two characteristics arrive from each side and the conditions at the discontinuity degenerate into

$$[u] = [p] = 0; \quad u = D,$$

i.e., the contact discontinuity propagates along the characteristic.

We can now return to the problem mentioned at the beginning, on extending the solution to the case of an arbitrary discontinuity.

4. The Break-Up of an Arbitrary Discontinuity

At some time t = 0 let the solution have an arbitrary discontinuity at point x = 0. Since we must find a solution in the immediate vicinity of the coordinate origin (more precisely, the asymptotics of the solution as $t \to 0$, $x \to 0$), it suffices to consider the case of piecewise constant initial conditions [1, 2, 6]

$$\left.\begin{aligned} \rho^-, u^-, \varepsilon^- \quad &\text{for} \quad x < 0; \\ \rho^+, u^+, \varepsilon^+ \quad &\text{for} \quad x > 0. \end{aligned}\right\} \quad (4.1)$$

The gasdynamics equations with initial conditions (4.1) can be solved as a function of $x/t = \xi$. We use the characteristic form of Eqs. (1.5) and convert from /48

x, t to variable ξ. We get a system of ordinary differential equations

$$\left.\begin{aligned} (u-\xi)\frac{dS}{d\xi} &= 0; \\ (u+c-\xi)\left(\frac{dp}{d\xi}+\rho c\frac{du}{d\xi}\right) &= 0; \\ (u-c-\xi)\left(\frac{dp}{d\xi}-\rho c\frac{du}{d\xi}\right) &= 0. \end{aligned}\right\} \qquad (4.2)$$

The initial conditions (4.1) now become the boundary conditions

$$(\rho, u, \varepsilon) \to (\rho^{\pm}, u^{\pm}, \varepsilon^{\pm}) \quad \text{for} \quad \xi \to \pm\infty. \qquad (4.3)$$

The fact that a boundary-value problem is stated for Eqs. (4.2), which form a first-order system should not seem strange, since we are not restricted here to continuous solutions. Possible discontinuities should as before satisfy conditions (3.1), where D denotes the value of ξ at which the discontinuity occurs. The discontinuity should satisfy not only condition (3.1) but also the correctness condition, whereby four characteristics must arrive at the discontinuity. As we remember, we have three families of characteristics, the slopes of which are u - c, u and u + c, respectively. Depending on the number of characteristics arriving from right and left, we distinguish three types of discontinuity. For a shock wave of the first type three characteristics arrive from the left and one from the right. This means that

$$\left.\begin{aligned} D &< (u-c)_l < u_l < u_l + c_l; \\ (u-c)_r &< D < u_r < u_r + c_r. \end{aligned}\right\} \qquad (4.4)$$

The next case is the contact discontinuity in which u and p are continuous and

$$\begin{aligned} u_l - c_l &< D = u_l < u_l + c_l; \\ u_r - c_r &< u_r = D < u_r + c_r. \end{aligned}$$

Finally, if we have a shock wave of the second type, one characteristic arrives from the left and three from the right:

$$\begin{aligned} u_l - c_l &< u_l < D < u_l + c_l; \\ u_r - c_r &< u_r < u_r + c_r < D. \end{aligned}$$

We turn to continuous solutions and classify them similarly. System of equations (4.2) is homogeneous with respect to the derivatives and, consequently, /49

will have a nontrivial solution only if its determinant is zero. The latter can occur in only three cases. The first is that of $(u-\xi) = 0$, when $dS \neq 0$ and $dp = du$. This is the previously considered case of a contact discontinuity. Further,

$$u - c = \xi \tag{4.5}$$

and here

$$\left.\begin{aligned} dS &= 0; \\ dp + \rho c du &= 0. \end{aligned}\right\} \tag{4.6}$$

The solution satisfying Eqs. (4.5) and (4.6) will be termed a rarefaction wave of the first type.

Finally, a rarefraction wave of the second type is obtained when

$$u + c = \xi$$

and

$$\begin{aligned} dS &= 0; \\ dp - \rho c du &= 0. \end{aligned}$$

Our problem is to construct a solution over the interval $-\infty < \xi < +\infty$ satisfying boundary conditions (4.3), using the above types of solutions as well as the trivial solutions $\rho, u, \varepsilon = \text{const}$ of system of equations (4.2).

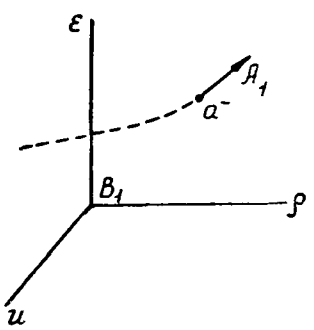

Figure 5.

The analysis henceforth is carried out in the ρ, u, ε space (Fig. 5). We have the point $a^-(\rho^-, u^-, \varepsilon^-)$. What points of the space can be reached from it traveling with a rarefaction wave of the first type? Obviously, these points form a curve satisfying Eq. (4.6) and passing through a^-. Variable ξ is a parameter along this curve and is defined by Eq. (4.5). At the point proper $a^- \xi = \xi^- = u^- - c^-$. Hence we may start moving from this point only when $\xi = \xi^-$, obviously, in the direction of increasing ξ, using the trivial solution $(\rho, u, \varepsilon) \equiv (\rho, u, \varepsilon)^-$ over the segment $-\infty < \xi < \xi^-$. The curve thus obtained is denoted by A_1. Let us now consider points which can be reached from a^- by traveling with a shock wave of the first type.

Conditions (3.1) at the discontinuity together with Eqs. (4.4) define the curve along which D is a parameter. This curve passes through a^-, since the discontinuity may also be

infinitesimal. Here, since

$$u_r - c_r < D < u_l - c_l,$$

then D at point a^- is equal to $u^- - c^- = \xi^-$ and decreases along the aforementioned curve, which is denoted by B_1. It is easy to show that A_1 and B_1 have a second-order tangency at point a^-. Thus, using solutions of the first type we can travel from point a^- to any point a_1 of curve $A_1 + B_1$, using the trivial solution $(\rho, u, \varepsilon) \equiv (\rho, u, \varepsilon)^-$ over segment $-\infty < \xi < \xi^-$ or $-\infty < \xi < D$. The set of points $A_1 + B_1$ cannot be expanded using solutions of the first type. In fact, let us, moving along curve A_1, reach point a, i.e., let us construct the solution to $\xi = \xi_1 = u_1 - c_1$. It is no longer possible to travel from a_1 to some point with a shock wave of the first type since, by virtue of Eqs. (4.4), this has to be done at $\xi = D < u_1 - c_1 = \xi_1$. Similar deliberation shows that one cannot move out from B_1 using solutions of the first type. /50

Further, drawing through each point a_1 of curve $A_1 + B_1$ the lines

$$p = \text{const}; \quad u = \text{const},$$

we get a surface whose points can be reached from a_1 using the contact discontinuity at $\xi_2 = u_1$. This can be done since $\xi_2 = u_1 > u_1 - c_1 = \xi_1$ when $a_1 \subset A_1$ and $\xi_2 = u_1 > D$ when $a_1 \subset B_1$.

Finally, we draw through each point of the surface thus obtained rarefraction waves of the second type in the direction of increasing $\xi = u + c$, and lines corresponding to shock waves of the second type in the other direction. All the problems which arise in the process are solved similarly to those above and we find that for any point $a^+(\rho^+, u^+, \varepsilon^+)$ the solution is generally obtained in three stages, i.e., the solution has the following form: a contact discontinuity on each side of which there is either one shock wave or one rarefaction wave. At segments between the contact discontinuity and the waves, as well as at the edges $(\xi \to \pm\infty)$ we have $\rho, u, \varepsilon = \text{const}$. Formulas giving the solution of the problem can be written for each specific form of function $p = p(\rho, \varepsilon)$.

5. The S. K. Godunov Method

Using the solution to the problem of break-up of an arbitrary discontinuity, one can construct the following original numerical method [6].

The x axis is broken up into short intervals Δx and the initial conditions at each such interval are replaced by constants, i.e., the functions are approximated by piecewise-constant functions. At the junction of each pair of intervals one must solve the problem of break-up of an arbitrary discontinuity, and when this solution is obtained one can obtain $\rho(x)$, $u(x)$ and $\varepsilon(x)$ at $t = \Delta t$ for each pair of /51

intervals. The time step Δt is selected so small that the solutions obtained for each two adjoining junctions do not interact. To carry out the following time step one must again have piecewise constant functions at layer $t=\Delta t$. This requires that the solution be somewhat averaged over intervals Δt. This can be done most naturally by conserving the mass, momentum and energy at each interval, i.e., by finding new average $\bar{\rho}$, u, $\bar{\varepsilon}$ from the formulas

$$\bar{\rho} = \frac{1}{\Delta x} \int \rho dx;$$

$$\bar{\rho}\bar{u} = \frac{1}{\Delta x} \int \rho u dx;$$

$$\bar{\rho}\left(\bar{\varepsilon} + \frac{\bar{u}^2}{2}\right) = \frac{1}{\Delta x} \int \rho \left(\varepsilon + \frac{u^2}{2}\right) dx,$$

where integration is carried out over each given interval.

In regions where the solution is smooth the values of ρ, u, ε for adjoining intervals differ little from one another and hence the formulas utilizing the solution of break-up of a discontinuity can be approximately replaced by simpler expressions. The latter can be obtained by carrying out a power series expansion of the former in $\rho^+ - \rho^-$, etc., and retaining the principal terms. The smoothness regions here are defined as those where the pressure gradient, for example, is smaller than some value.

Under the above method shock waves are not computed separately, but are rather calculated in the form of narrow zones (with a width of several computational intervals) with large variable gradients. This method has its variation in which the shock wave is not "smeared out."

6. Difference Methods

There exists an extensive class of numerical methods of solution of differential equations which, by virtue of the method by which they are obtained, are called finite-difference or simply difference methods. They are constructed in general as follows. Firstly, a computational grid is selected, i.e., one selects points /5
x_k, t^n at which the unknown functions (ρ_k^n, etc.) are to be computed. Secondly, the derivatives contained in the equations are replaced by finite differences. This yields a system of algebraic difference equations for the unknown ρ_k^n, etc., in which the grid-mesh spacings serve as parameters. And thirdly, if it is necessary, an algorithm for solving the algebraic system thus obtained is worked out.

All the difference methods are approximate, yielding a solution with some error. It is of principal importance to know whether this error can be made as small as desired and whether the solution of the difference equations converges to the exact solution as the lattice spacings are reduced.

The theory of difference methods is as yet the theory of linear difference methods for solving linear differential equations. We shall show schematically how this theory treats the problem of convergence [4, 5].

It is required to solve the problem

$$LU = F, \tag{6.1}$$

where L is a linear operator, U is the sought solution and F are known functions (initial conditions, etc.). Problem (6.1) cannot be solved, i.e., operator L^{-1} is unknown and hence, instead of (6.1), we solve the problem

$$L_h U_h = F_h, \tag{6.2}$$

where h is some variable (usually the lattice spacing). In order that $U_h \to U$ as $h \to 0$ one must satisfy the two following conditions.

Firstly it is required that Eq. (6.2) approximate Eq. (6.1), i.e.,

$$L_h U \to LU; \quad F_h \to F \quad \text{for} \quad h \to 0,$$

and secondly, that operator L_h^{-1} be uniformly bounded over h, i.e., that the difference method be, so to speak, stable.

In fact,

$$U_h - U = L_h^{-1} L_h (U_h - U) = L_h^{-1} (L_h U_h - LU + LU - L_h U) = L_h^{-1} (F_h - F + LU - L_h U).$$

Since both the differences in parentheses approach zero as $h \to 0$, and operator L^{-1} is bounded, then $U_h - U \to 0$ as $h \to 0$. The above delibration obviously does not prove the convergence theorem, since we have not defined the classes of functions acted upon by operators, we have not defined the norm used for estimating functions, etc. /53

We now describe some difference schemes. We return to the system of equations in the form of (1.4), written in Lagrangian coordinates. Due to the specific form of the system, we can construct for it the following difference scheme. Quantities ρ and ε will be calculated at points s_k, t^n and be denoted ρ_k^n, ε_k^n. Here $s_{k+1} - s_k = h$, $t^{n+1} - t^n = \tau$. Velocity u will be calculated at points $s_{n+1/2} = s_k + h/2$, $t^{n+1/2} = t^n + \tau/2$. In this "checked" grid, system of equations (1.4) is replaced naturally by the difference scheme (the "cross" scheme):

$$\left.\begin{aligned}
&\frac{\rho_k^{n+1}-\rho_k^n}{\tau}+(\rho_k^n)^2\,\frac{u_{k+\frac{1}{2}}^{n+\frac{1}{2}}-u_{k-\frac{1}{2}}^{n+\frac{1}{2}}}{h}=0;\\
&\frac{u_{k+\frac{1}{2}}^{n+\frac{1}{2}}-u_{k+\frac{1}{2}}^{n+\frac{1}{2}}}{\tau}+\frac{p_{k+1}^{n+1}-p_k^{n+1}}{h}=0;\\
&\frac{\varepsilon_k^{n+1}-\varepsilon_k^n}{\tau}+p_k^n\,\frac{u_{k+\frac{1}{2}}^{n+\frac{1}{2}}-u_{k-\frac{1}{2}}^{n+\frac{1}{2}}}{h}=0.
\end{aligned}\right\} \qquad (6.3)$$

The fact that Eqs. (6.3) approximate Eqs. (1.4) is obvious. We now consider the question of stability of Eqs. (6.3), more precisely, of this system's linear model. The latter is obtained by regarding coefficients ρ^2 in the first and p in the third of the equations constant, and by replacing the difference Δp in the second equation by $p_\rho \Delta\rho + p_\varepsilon \Delta\varepsilon$ with constant p_ρ and p_ε. The linear system thus obtained is a linear operator, transforming ρ^n, $u^{n+\frac{1}{2}}$, ε^n into ρ^{n+1}, $u^{n+\frac{1}{2}}$, ε^{n+1}. We denote it by L_τ^{-1}. We now must estimate the norm of the operator transforming the initial conditions (t = 0) into a solution at some given finite time (for example, t = 1). Obviously, it is equal to the norm of operator L_τ^{-1} to power $1/\tau$ and we will have stability if the norm of L_τ^{-1} is smaller than or equal to unity. The latter is estimated as the maximum absolute value of the operator's eigenvalues. The eigenvalue of operator L_τ^{-1} is

/54

$$\left.\begin{aligned}
\rho_k &= \rho_0 e^{ik\varphi};\\
u_k &= u_0 e^{ik\varphi};\\
\varepsilon_k &= \varepsilon_0 e^{ik\varphi}
\end{aligned}\right\} \qquad (6.4)$$

with arbitrary φ.

We denote the eigenvalues by λ and substitute Eqs. (6.4) into Eqs. (6.3), upon linearization of the latter. Dropping the indices of coefficients, we get

$$\frac{\lambda-1}{\tau}\rho_0+\rho^2\,\frac{e^{\frac{i\varphi}{2}}-e^{-\frac{i\varphi}{2}}}{h}u_0=0;$$

$$\frac{\lambda-1}{\tau}u_0+\lambda\left(p_\rho\frac{e^{\frac{i\varphi}{2}}-e^{-\frac{i\varphi}{2}}}{h}\rho_0+\rho_\varepsilon\frac{e^{\frac{i\varphi}{2}}-e^{-\frac{i\varphi}{2}}}{h}\varepsilon_0\right)=0;$$

$$\frac{\lambda-1}{\tau}\varepsilon_0+p\frac{e^{\frac{i\varphi}{2}}-e^{-\frac{i\varphi}{2}}}{h}u_0=0.$$

This system has the nontrivial solution ρ_0, u_0, ε_0 if its determinant is equal to zero, i.e.,

$$\begin{vmatrix} \frac{\lambda-1}{\tau} & \rho^2\frac{2i\sin\frac{\varphi}{2}}{h} & 0 \\ \lambda p_\rho\frac{2i\sin\frac{\varphi}{2}}{h} & \frac{\lambda-1}{\tau} & \lambda p_\varepsilon\frac{2i\sin\frac{\varphi}{2}}{h} \\ 0 & p\frac{2i\sin\frac{\varphi}{2}}{h} & \frac{\lambda-1}{\tau} \end{vmatrix}=0.$$

whereby it is easily found that one eigenvalue is equal to unity and the two others are found from

$$\lambda+\frac{1}{\lambda}=2-4\frac{\tau^2}{h^2}\rho^2c^2\sin^2\frac{\varphi}{2}. \qquad (6.5)$$ /55

If $\lambda=\lambda_0$ satisfies this equation, then $\lambda=1/\lambda_0$ also satisfies it. Also, if $|\lambda_0|<1$, then $|\frac{1}{\lambda_0}|>1$. Hence, the only suitable case for our purposes is $|\lambda|=1$, i.e., $\lambda=e^{i\theta}$. Substituting this expression in Eq. (6.5), we have

$$\cos\theta=1-\frac{2\tau^2}{h^2}\rho^2c^2\sin^2\frac{\varphi}{2}.$$

Consequently, $|\lambda|=1$, if the latter expression is real and is contained between -1 and +1. This yields the stability condition

$$\frac{\tau}{h}\rho c<1. \qquad (6.6)$$

Obviously, the above scheme and its examination have meaning only for smooth solutions. To calculate discontinuous solutions the method should be

changed to make allowance for relationships on the shock wave. However, difference schemes for calculating flows with shock waves may be constructed also in a different way.

7. Smearing Out of Shock Waves

As is known, discontinuous solutions of a system of equations can be regarded as the limit of smooth solutions of another, distrubed system, as the disturbing variable approaches zero. The system of gasdynamic equations presented in Sec. 1 makes no allowance for various dissipative effects — viscosity, heat conduction, etc. Hence, the natural way of introducing these disturbances into the system is to make an allowance for these effects, the viscosity, for example. But the actual physical viscosity is, as a rule, extremely small, so that the smooth solution obtained by using it can be practically regarded as smooth, i.e., the width of the zone of smearing out of the shock wave is found to be too narrow and an exceedingly fine lattice spacing is needed for calculations. However, the width of this zone can be increased artificially, by taking a sufficiently large viscosity coefficient.

Formally the introduction of viscosity involves replacing p in the equations by $\Pi = p + q$, where q is the viscosity. Von Neumann suggested that this be done in the form [3, 4]

$$q = a^2 h^2 \rho \left(\frac{\partial u}{\partial s}\right)^2$$

for $\frac{\partial u}{\partial s} < 0$ and $q = 0$ in the opposite case. Here a is a constant and h is the lattice spacing of the difference grid. /56

Let us see how much viscosity smears out a shock wave moving at constant velocity D (the solution to the left and right of the discontinuity is constant).

The system of equations has the form

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial s} = 0; \qquad \frac{\partial u}{\partial t} + \frac{\partial \Pi}{\partial s} = 0; \\ &\frac{\partial \varepsilon}{\partial t} + \Pi \frac{\partial u}{\partial s} = 0; \qquad \Pi = p + a^2 h^2 \rho \left(\frac{\partial u}{\partial s}\right)^2. \end{aligned}\right\} \tag{7.1}$$

We seek the solution in the form of a function of

$$\xi = s - Dt.$$

Transforming from s, t to ξ, we get instead of Eqs. (7.1) the system

$$-D\rho' + \rho^2 u' = 0; \quad -Du' + \Pi' = 0; \quad -D\varepsilon' + \Pi u' = 0;$$
$$\Pi = \rho + a^2 h^2 \rho (u')^2 .$$

Integration of the first three equations yields

$$\left.\begin{aligned} &\frac{D}{\rho} + u = C_1; \\ &-Du + \Pi = C_2; \\ &-D\varepsilon + C_2 u + D\frac{u^2}{2} = C_3; \\ &\Pi = p + a^2 h^2 \rho (u')^2 , \end{aligned}\right\} \qquad (7.2)$$

where C_1, C_2 and C_3 are constant. It can be shown that the constants $(\rho, u, \varepsilon)_l$ and $(\rho, u, \varepsilon)_r$, equal to the values to the left and right of the shock wave, satisfy Eqs. (7.2).

Eliminating ρ and ε from Eqs. (7.2), we get an expression for $u(\xi)$, integration of which from ξ_l to ξ_r, corresponding to u_l or u_r, yields the width $\xi_l - \xi_r$ of the smearing out zone and the solution for $\xi_l < \xi < \xi_r$. Thus, if $p = (\gamma - 1)\rho\varepsilon$, then

$$\xi_r - \xi_l = \pi \sqrt{\frac{2}{\gamma + 1}} \cdot ah,$$

i.e., the wave is smeared out over a given number of lattice spacings of the difference grid.

It is very important that the width of the smearing out zone not depend on the specific values of the quantities to the left and right of the shock wave and on the latter's velocity.

A "cross" type system such as given by Eqs. (6.3) can also be constructed /57
for the system with viscosity (7.1), replacing Π_k^n by the expression

$$\Pi_k^n = p_k^n + a^2 \rho_k^n \left(u_{k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{k-\frac{1}{2}}^{n-\frac{1}{2}} \right)^2 .$$

This viscosity can manifest itself only in the shock wave region, where its role is predominant. It can be said that the shock wave is smeared out because the equations in this zone are parabolic in nature and the process is qualitatively

determined by the equation

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial s} = 0, \quad q = a^2 h^2 \rho \left(\frac{\partial u}{\partial s}\right)^2 . \tag{7.4}$$

Hence in order to find stability conditions, we restrict ourselves to consideration, in difference equations (6.3) and (7.3), of terms corresponding to Eq. (7.4). Linearizing the equation, we get instead of Eq. (7.4)

$$\frac{\partial u}{\partial t} - A \frac{\partial^2 u}{\partial s^2} = 0, \quad A = 2a^2 h^2 \rho \left|\frac{\partial u}{\partial s}\right| .$$

Reverting to the difference analog of the latter equations (in accordance with Eqs. (6.3) and (7.3)), we consider, as before, the question of stability using eigenfunctions. We get for λ the expression

$$\frac{\lambda - 1}{\tau} = A \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{h^2} .$$

Consequently, $|\lambda| \leq 1$, if

$$\frac{\tau A}{h^2} < \frac{1}{2} ,$$

or, substituting the expression of A and replacing $\frac{\partial u}{\partial s}$ by $\frac{1}{h} \Delta u$, we get the stability condition

$$4a^2 \rho |\Delta u| \frac{\tau}{h} < 1. \tag{7.5}$$

8. Implicit Schemes

The above difference scheme is classified as the so-called explicit scheme. In it the unknown quantities of the upper (n + 1)th layer are explicitly expressed in terms of quantities of the bottom nth layer. They are also called local schemes, since in using them for calculating the unknown ρ_k^{n+1}, etc., one need only know the values of ρ_k^n, etc., at the closest two or three points. For this reason all the local schemes are stable only for a given lattice ratio spacing, since the grid points used should cover the dependence domain of the point, where the values of the function are calculated. The main advantages of the explicit, local schemes is their logical simplicity and the small volume of computation formulas of all the schemes. However, this does not always result in a minimum volume of computations. For

/58

example, if we select a three-dimensional lattice spacing from accuracy considerations, then the time step required by stability conditions is frequently much smaller than accuracy requires. It is hence natural to try to remove the limitations imposed on the lattice spacing by stability conditions. This is obtained implicit schemes, which shall now be considered [4-7].

Stability condition (7.5) appears as the result of the local method of replacing the differential expression contained in the viscosity with its difference counterpart. Hence we now replace $\frac{\partial u}{\partial s}$ contained in q, using the difference on the upper layer. Namely, we write instead of (7.3) the expression

$$\Pi_k^n = p_k^n + a^2 \rho_k^n \left(u_{k+1/2}^{n-1/2} - u_{k-1/2}^{n-1/2} \right) \left(u_{k+1/2}^{n+1/2} - u_{k-1/2}^{n+1/2} \right). \tag{8.1}$$

In this way we get a system of linear algebraic equations for determining all the $u^{n+3/2}$. The characteristic form of this system, consisting in the fact that each equation contains only three consecutive values $u_{k-1/2}^{n+3/2}$, $u_{k+1/2}^{n+3/2}$, $u_{k+3/2}^{n+3/2}$, makes it possible to use the quite effective sweep method for solving it. The substance of this method consists in the following. We write each equation of the system in the form

$$A_{k+1/2} u_{k-1/2} + B_{k+1/2} u_{k+1/2} + C_{k+1/2} u_{k+3/2} + D_{k+1/2} = 0, \tag{8.2}$$

where A, B, C and D are known quantities and the superscript n + 3/2 has been dropped. Let the unknown $u_{k-1/2}$, $u_{k+1/2}$ be related by the expression /59

$$u_{k-1/2} = x_k u_{k+1/2} + y_k. \tag{8.3}$$

Eliminating $u_{k-1/2}$ from Eqs. (8.2) and (8.3), we get

$$u_{k+1/2} = -\frac{C_{k+1/2}}{A_{k+1/2} x_k + B_{k+1/2}} u_{k+3/2} - \frac{D_{k+1/2} + A_{k+1/2} y_k}{A_{k+1/2} x_k + B_{k+1/2}}. \tag{8.4}$$

Equation (8.4) can be written in the form of (8.3) by setting

$$\left.\begin{aligned} x_{k+1} &= -\frac{C_{k+\frac{1}{2}}}{A_{k+\frac{1}{2}}x_k + B_{k+\frac{1}{2}}}; \\ y_{k+1} &= -\frac{D_{k+\frac{1}{2}} + A_{k+\frac{1}{2}}y_k}{A_{k+\frac{1}{2}}x_k + B_{k+\frac{1}{2}}}, \end{aligned}\right\} \qquad (8.5)$$

but we take k larger by unity. Thus, if we write the left boundary condition in the form of Eq. (8.3), then recurrent formulas (8.5) can be used for eliminating in sequence all the u from the system and obtaining Eq. (8.3) for the extreme right values of u. Solving the latter together with the relationship expressing the right boundary condition, we get the value of u at the right edge of the computational region. Then, since all the x_k and y_k are known, all the u are found in sequence from Eq. (8.3).

Investigating the difference scheme thus obtained for stability, we will get, as in the preceding section, for the eigenvalue λ the expression

$$\frac{\lambda - 1}{\tau} - \lambda A \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{h^2} = 0$$

(see Eqs. (7.4) and (8.1)), whence

$$\lambda = \left[1 + \frac{2\tau A}{h^2}(1 - \cos\varphi)\right]^{-1},$$

i.e., $|\lambda| \leq 1$ for all the φ and, consequently, the second of the stability conditions (7.5) can be removed from the above computational scheme.

One may take another step and remove the first stability condition (6.6), using completely the implicit scheme. For our system (7.1) of equations with viscosity, one can suggest the following scheme: /60

$$\left.\begin{aligned}
&\frac{\rho_k^{n+1}-\rho_k^n}{\tau}+(\rho_k^n)^2\frac{u^{n+1}_{k+\frac{1}{2}}-u^{n+1}_{k-\frac{1}{2}}}{h}=0;\\
&\frac{u^{n+1}_{k+\frac{1}{2}}-u^{n}_{k+\frac{1}{2}}}{\tau}+\frac{\Pi^{n+1}_{k+1}-\Pi^{n+1}_{k}}{h}=0;\\
&\frac{\varepsilon_k^{n+1}-\varepsilon_k^n}{\tau}+\Pi_k^n\frac{u^{n+1}_{k+\frac{1}{2}}-u^{n+1}_{k-\frac{1}{2}}}{h}=0;\\
&\Pi_k^{n+1}=p_k^{n+1}+a^2\rho_k^n(u^n_{k+\frac{1}{2}}-u^n_{k-\frac{1}{2}})(u^{n+1}_{k+\frac{1}{2}}-u^{n+1}_{k-\frac{1}{2}}).
\end{aligned}\right\}\qquad(8.6)$$

The above system of difference equations can also be solved by the sweep method, first replacing them with a linear system. If it is assumed that ρ_k^{n+1}, $u^{n+1}_{k+\frac{1}{2}}$, p_k^{n+1}, are the unknowns, then nonlinearity will exist only due to presence of the term ε_k^{n+1}. Replacing $p=p(\rho, \varepsilon)$ by an equation of state in the form $\varepsilon=\varepsilon(p,\rho)$, we express ε_k^{n+1} as

$$\varepsilon_k^{n+1}=\varepsilon_k^n+(\varepsilon_p)_k^n(p_k^{n+1}-p_k^n)+(\varepsilon_\rho)_k^n(\rho_k^{n+1}-\rho_k^n).\qquad(8.7)$$

The system of equations in (8.6) and (8.7) is linear. Replacing ρ_k^{n+1} and p_k^{n+1} with their expressions in terms of u^{n+1} (the first and third of equations (8.6) and (8.7)), we get difference equations of the form of (8.2), which can be solved by the sweep method.

The stability of the above implicit difference scheme for any ratio of lattice spacings τ and h can be established by the usual method.

9. Two-Dimensional Schemes

We now consider methods of calculating two-dimensional gasdynamic problems, i.e., problems with two space variables. Many of the one-dimensional methods /61 can be generalized more or less obviously to include the two-dimensional case, and this is also true of the smearing out of shock waves. We shall not consider here the two-dimensional versions of various known schemes, but shall rather consider problems typical of two-dimensional (and, in general, of multidimensional) problems.

In one-dimensional problems the computational region is always a straight-line segment (segment of the x or s axis). In two-dimensional problems this region may be of many forms. Hence, if we do not wish to consider the quite complicated questions of approximation of boundaries and boundary conditions in a specified rectangular difference grid, we must carry out calculations in curvilinear coordinates, mapping the computational region onto a rectangle. Here the space variables x and y become functions of new coordinates, which are denoted by α and β, and of time t if the coordinate system moves. By virtue of the same fact, the substitution of variables imposes limitations on the difference grid, i.e., functions $x(\alpha, \beta, t)$ and $y(\alpha, \beta, t)$ should not vary sharply, since otherwise the computational accuracy will suffer.

Many problems can be computed in Lagrangian coordinates, i.e., by specifying $x = x(\alpha, \beta, 0)$ and $y = y(\alpha, \beta, 0)$ one finds x and y from the expressions

$$\frac{\partial x}{\partial t} = u; \quad \frac{\partial y}{\partial t} = v,$$

where u and v are components of the velocity vector along x and y, respectively. However, unlike the one-dimensional case, two points $\alpha_1\beta_1$ and $\alpha_2\beta_2$, initially close to one another and hence having close Lagrangian coordinates, may find themselves quite removed after time has elapsed. This, for example, occurs for points situated in the vicinity of a contact discontinuity, along which slip of one gas layer relative to another is possible. If the irregularity in mass motion reduces to this slip, then the so-called Eulerian-Lagrangian coordinates are satisfactory. These are introduced as follows.

Let α be a Lagrangian and β be an Eulerian coordinate and equal for simplicity to y,

$$x = x(\alpha, \beta, t'); \quad y = \beta; \quad t = t'.$$

The region is initially subdivided into layers with α = const and each such layer retains its α coordinate in time which is the fact making the coordinate α Lagrangian. /62

We introduce an expression for finding $x(\alpha, \beta, t)$. We represent in Fig. 6 the line $\alpha = \alpha_0$ at two close times t and $t + \Delta t$. The gas particle will travel during time Δt from point M_1 to M_2. Vector M_1M_2 is defined as $u\Delta t, v\Delta t$. Since points M_1 and M_2 correspond to $\alpha_0\beta_0$ during different time instants, $\frac{\partial x}{\partial t} \cdot \Delta t$ is the distance between M_1 and M_3. Since the slope of the lines $\alpha = \alpha_0$ is defined as $\frac{\partial x}{\partial \beta}$, it is easy to obtain

$$\frac{\partial x}{\partial t} \cdot \Delta t = u\Delta t - v\Delta t \frac{\partial x}{\partial \beta}$$

or

$$\frac{\partial x}{\partial t'} + v\frac{\partial x}{\partial \beta} = u$$

i.e., an equation for finding $x(\alpha, \beta, t')$ and consequently also of x_α and x_β which will enter the coefficients of the system of equations after substitution of variables.

The above example shows that frequently the question of selecting a coordinate system is far from being the most trivial of all the questions in constructing a numerical method.

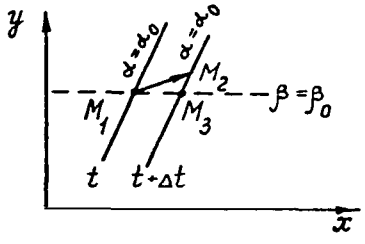

Figure 6.

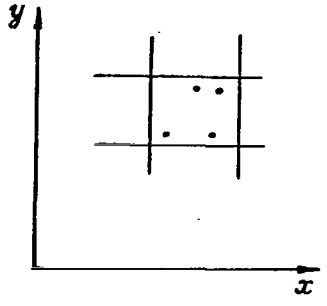

Figure 7.

Below we shall present two methods in which the problem of the difference grid is partially or totally removed, while the advantages of the Lagrangian coordinate system are retained.

We write the system of gasdynamics equations in the form [1]

/63

$$\left.\begin{aligned}
&\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0;\\
&\frac{du}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0;\\
&\frac{dv}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0;\\
&\frac{d}{dt}\left(\varepsilon + \frac{u^2+v^2}{2}\right) + \frac{1}{\rho}\frac{\partial pu}{\partial x} + \frac{1}{\rho}\frac{\partial pv}{\partial y} = 0,
\end{aligned}\right\} \qquad (9.1)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ is a derivative along the particle path.

Harlow [8] suggested the following original computational method. The space to be computed is subdivided in the x, y plane by a rectangular mesh (Fig. 7). Some (usually small) number of gas particles is contained in each mesh cell. The mass of each particle is prescribed. Density ρ in a given cell is defined as the ratio of the total mass of particles of this cell to the volume (area) of the cell. In addition, velocities u and v and the internal energy ε are defined for each cell. Since the law of conservation of mass is satisfied automatically, the first of equations (9.1) is not used. Using the remaining three of equations (9.1) we naturally find new (not final) values $\tilde{u}$, $\tilde{v}$, $\tilde{\varepsilon}$. For example, $\frac{\partial u}{\partial t}$ is replaced by $\frac{\tilde{u}-u}{\Delta t}$ and $\frac{\partial p}{\partial x}$ is replaced by $\frac{\Delta_x p}{2\Delta x}$, where $\Delta_x p$ is the difference in the p of right and left cells and Δx is the mesh spacing. The remaining terms of Eqs. (9.1) are treated similarly. From the $\tilde{u}$, $\tilde{v}$, $\tilde{\varepsilon}$ obtained we calculate the total momentum $M\tilde{u}$, $M\tilde{v}$ of the cell (M is the cell mass) and the total energy $M\left(\tilde{\varepsilon}+\frac{\tilde{u}^2+\tilde{v}^2}{2}\right)$. Then new coordinates $x+\tilde{u}\Delta t$, $y+\tilde{v}\Delta t$ of each particle are found. If then the particle moves into the neighboring cell, it takes with it its portion of mass, momentum and energy. Then the mass, momentum and total energy of each cell are recalculated and these are used for obtaining new, final values of ρ, u, v and ε.

We have described the substance of Harlow's method without dwelling on its variations (for example, for the case of a mixture of two gases). Results obtained by this method show that it is universal and reliable in spite of the fact that it does not have a rigorous theoretical basis. /64

In closing we consider still another method of particles [9]. We shall dispense with any regular computational grid and assume that the values of unknown functions ρ, u, v, ε are calculated at some points x, t, i.e., gas particles, located at the given time t arbitrarily in the x, y plane. In order to obtain a method for calculating the x, t, ρ, u, v, ε of each particle during the next time instant $t+\Delta t$, we proceed as follows. Since Δt is small, the coefficients of the system of equations for this time interval will change little and in the vicinity of the given particle they can be regarded as constant. Consequently, to obtain an approximate solution the system can be replaced by a linear system. For a linear hyperbolic system with constant coefficients one may write formulas expressing the values of functions, i.e., the solutions at time $t+\Delta t$ in terms of integrals of these functions at time t (similar to the Poisson formula for the wave equation). Using adjacent (at the given time) points ("particles") as reference points for interpolation of the integrands u, v, ρ, ε, one may obtain the given quadrature formulas. The latter are the difference formulas needed by us, yielding the values at $t+\Delta t$. The new values of coordinates of the particles are obtained as $x+u\Delta t$, $y+v\Delta t$.

A problem of some importance in this method is that of finding a good means of selecting "adjacent" points: what points should be used for computing the given point and what points should not be used. But this question is not discussed here.

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THE "DECOMPOSITION" METHOD IN SOLVING PROBLEMS OF MATHEMATICAL PHYSICS

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A difference scheme must satisfy not only the ordinary approximation and /6
stability requirements, but a number of additional requirements such as economy, high accuracy, divergence, etc. Hence construction of difference schemes on the basis of ordinary homogeneous approximations has involved various difficulties which have been resolved by using the method of small steps, utilizing inhomogeneous approximation.

The first results in this direction were obtained by Peaceman, Rachford and Douglas [1-3]. Subsequently these results were developed and enlarged upon in depth in [4-8]. All this work is based on the Peaceman, Rachford and Douglas schemes (alternating direction methods (ADM) and the Douglas-Rachford stabilizing correction method). These methods require approximation of the differential equation either at each small step (alternating direction method), or at the first small step with subsequent stability correction (stabilizing correction method).

A large number of studies by American workers is concerned with optimal selection of parameters of iteration schemes [9-11] and with obtaining difference schemes of a higher order of accuracy [12].

Investigations by Soviet workers are based on decomposition of complex operators into simpler ones. Under this approach integration of a given equation reduces to successive integration of equations with a simpler structure. Here the small-steps schemes need satisfy the approximation and stability conditions only in the final result. This allows for flexible construction of schemes for substantially all the principal equations of mathematical physics.

One of the first works in this direction was devoted to application of the de- /6
composition method to problems of multi-dimensional hydrodynamics, in which explicit decomposition schemes were used.

Yanenko [14] suggested a method of small steps based on breaking up of the multidimensional heat conduction equation into a sequence of elementary one-dimensional equations. The method of small steps was further developed in [15-17].

A sequence of studies was carried out by Samarskiy in conjunction with investigation of locally one-dimensional schemes, which is closely related to the decomposition method. He also considered schemes of higher order of accuracy for heat conduction equations. Special methods of a priori estimates were developed by him for the study of convergence [18-21].

An algorithm for constructing an implicit scheme of difference equations with a factored matrix was given in [22]. Yanenko [23] pointed out a method for augmenting the local difference operator to a factored operator (approximate factoring of the difference operator).

D'yakonov [24-26] formulated a method for constructing schemes with decomposable (factored) upper operator for parabolic and hyperbolic type equations.

Decomposition methods for approximate integration of irregular systems in dynamic meteorology (weather forecasting problems) were suggested in [27-29].

Decomposition methods were found quite effective in solving multi-dimensional kinetic Boltzmann equations [30-31], equations of the theory of elasticity [32-34], etc.

It was found possible to treat the decomposition method as a method of weak approximation, which allowed one to use it for investigation of the Cauchy problem for correctness.

Experience accumulated in solving complex problems of mathematical physics by decomposition methods shows that they are effective and universal. Various aspects of the decomposition method are presented below.

1. Methods of Decomposition of Steady-State Problems

We consider the steady-state problem

$$A\varphi = f, \tag{1.1}$$

where A is some operator (we assume for simplicity that it is linear and symmetrical), f is a known function, and φ is the sought solution. It is assumed that the domains of definition of given functions f and φ are in real vector space with a scalar product and corresponding norm. /68

We construct the relaxation process

$$\varphi^{j+1} = \varphi^{j} - \tau(A\varphi^{j} - f), \tag{1.2}$$

where τ is an arbitrary relaxation variable. The iteration process is now written in the form

$$\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + A\varphi^{j} = f. \tag{1.3}$$

It is easy to see that if iteration (1.3) converges, then the limiting element φ^{∞} is a solution of problem (1.1). The identity transform of Eqs. (1.2) and (1.3) is

$$\varphi^{j+1} = T\varphi^{j} + \tau f,$$

where T is an operator with step

$$T = E - \tau A.$$

In order for iteration (1.2) to converge it suffices that the norm of operator T satisfy the condition

$$\|T\| < 1.$$

In many cases it is possible to select τ so as to obtain the optimal convergence of the process.

We now consider the more general iteration process

$$B\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + A\varphi^{j} = f, \tag{1.4}$$

where B as yet is an arbitrary operator. If we define it as $B = \tau A$, then by means of a single iteration we get the exact solution of the problem

$$\varphi^{j+1} = A^{-1}f.$$

However, determination of A^{-1} is no easier than solution of Eq. (1.1), meaning that this generality in constructing the iteration process is useless here.

We consider a particular case of operator B

$$B = \prod_{\alpha=1}^{n}\left(E + \frac{\tau_\alpha}{2}A_\alpha\right),$$

where A_α are some arbitrary operators of a structure simpler than A, while τ_α are arbitrary constants. Then iteration process (1.4) takes the form /69

$$\prod_{\alpha=1}^{n}\left(E + \frac{\tau_\alpha}{2}A_\alpha\right)\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + A\varphi^{j} = f. \tag{1.5}$$

If

$$A = \sum_{\alpha=1}^{n} A_\alpha,$$

$$A_\alpha = A_\alpha \quad (\alpha = 1, 2, \ldots, n),$$

we arrive at a special form of the universal algorithm

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau_\alpha}{2}A_\alpha\right)\frac{\varphi^{j+1}-\varphi^{j}}{\tau}+\sum_{\alpha=1}^{n}A_\alpha\varphi^{j}=f. \quad (1.6) \qquad (1.6)$$

We note that iteration process (1.6) has n + 1 arbitrary constants $\tau_1, \tau_2, \ldots, \tau_n, \tau$, which can be expressed in such a manner that the iteration process will converge most rapidly. System of equations (1.6) was regarded by Douglas and Gunn, Samarskiy, D'yakonov, Yanenko and others as a scheme for generating various methods of implementation of the decomposition method. It was shown by a number of studies that all the known decomposition methods possessing the property of total approximation reduce to schemes (1.5) or (1.6). And if this is so, then the question arises of constructing a scheme for implementing the iteration process. In particular, we will have for scheme (1.5)

$$\left.\begin{aligned}
&\left(E+\frac{\tau_1}{2}A_1\right)\psi^{j+1/n}=F^{j};\\
&\left(E+\frac{\tau_2}{2}A_2\right)\psi^{j+2/n}=\psi^{j+1/n};\\
&\cdots\cdots\cdots\cdots\cdots\\
&\left(E+\frac{\tau_n}{2}A_n\right)\psi^{j+1}=\psi^{j+\frac{n-1}{n}};\\
&\varphi^{j+1}=\varphi^{j}+\tau\psi^{j+1},
\end{aligned}\right\} \qquad (1.7)$$

where

$$F^{j}=-A\varphi^{j}+f$$

is the discrepancy of the iteration process.

2. Methods of Separation of Unsteady Problems /70

We consider the unsteady problem

$$\frac{\partial\varphi}{\partial t}+A\varphi=f.$$

whose solution is now sought in the form

$$\frac{\varphi^{j+1}-\varphi^{j}}{\tau}+A\varphi^{j}=f^{j+1/2}. \qquad (2.1)$$

This is an explicit scheme of calculation with a first-order approximation and, as a rule, with large stability limitation. Instead of Eq. (2.1) let us consider the implicit scheme

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}A_\alpha\right)\frac{\varphi^{j+1}-\varphi^j}{\tau}+A\varphi^j=f^{j+1/2},$$

where $\tau=\Delta t$, while A_α satisfies the condition

$$\sum_{\alpha=1}^{n}A_\alpha=A+0(\tau).$$

In the particular case we have

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\frac{\varphi^{j+1}-\varphi^j}{\tau}+\sum_{\alpha=1}^{n}\Lambda_\alpha\varphi^j=f^{j+1/2}, \tag{2.2}$$

where

$$\Lambda=\sum_{\alpha=1}^{n}\Lambda_\alpha.$$

This is a scheme of a general purpose algorithm, the implementation of which is again given by a system of equations such as (1.7). It can be shown that difference scheme (2.2) has a second order of accuracy with respect to τ on smooth functions. In fact, let us apply the operators in (2.2). This yields

$$\left(E+\frac{\tau}{2}\Lambda+\frac{\tau^2}{4}R\right)\frac{\varphi^{j+1}-\varphi^j}{\tau}+\Lambda\varphi^j=f^{j+1/2}, \tag{2.3}$$

where

$$R=(\Lambda_1\Lambda_2+\ldots+\Lambda_{n-1}\Lambda_n)+\ldots+\left(\frac{\tau}{2}\right)^{n-2}\Lambda_1\Lambda_2\ldots\Lambda_n.$$

Equation (2.3) is written in the form

$$\frac{\varphi^{j+1}-\varphi^j}{\tau}+\Lambda\frac{\varphi^{j+1}+\varphi^j}{2}=f^{j+1/2}-\frac{\tau^2}{4}R\left(\frac{\varphi^{j+1}-\varphi^j}{\tau}\right).$$

Since expression

$$R\frac{\varphi^{j+1}-\varphi^{j}}{\tau}$$

is bounded as $\tau \to 0$, we have proved our assertion.

Let us now prove the stability of this universal algorithm. We restrict ourselves to the case where operators Λ and Λ_α are commutative and have a positive spectrum and a complete system of eigenfunctions. The solution of the problem is sought in the form

$$\varphi_k^j = \eta_k^j \psi_1^{(k)} \psi_2^{(k)} \cdots \psi_n^{(k)}, \tag{2.4}$$

where $\psi_\alpha^{(k)}$ are the eigenfunctions of the homogeneous problems

$$\Lambda_\alpha \psi_\alpha^{(k)} = \lambda_\alpha^{(k)} \psi_\alpha^{(k)}.$$

Substitution of Eq. (2.4) into Eq. (2.2) for $f^{j+1/2} = 0$ yields

$$\prod_{\alpha=1}^{n}\left(1+\frac{\tau}{2}\lambda_\alpha^{(k)}\right)\frac{\eta_k^{j+1}-\eta_k^{j}}{\tau}+\lambda^{(k)}\eta_k^{j}=0, \tag{2.5}$$

where

$$\lambda^{(k)} = \lambda_1^{(k)} + \lambda_2^{(k)} + \cdots + \lambda_n^{(k)}.$$

Equation (2.5) is solved for η_k

$$\eta_k = \frac{1-\lambda^{(k)}\frac{\tau}{2}+\left(\frac{\tau}{2}\right)^2 R_\alpha^{(k)}}{1+\lambda^{(k)}\frac{\tau}{2}+\left(\frac{\tau}{2}\right)^2 R_\alpha^{(k)}}, \tag{2.6}$$

where

$$R_\alpha^{(k)} = (\lambda_1^{(k)}\lambda_2^{(k)} + \cdots + \lambda_{n-1}^{(k)}\lambda_n^{(k)}) + \cdots + \left(\frac{\tau}{2}\right)^{n-2}\lambda_1^{(k)}\lambda_2^{(k)}\cdots\lambda_n^{(k)}.$$

It is seen from Eq. (2.6) that when $\tau > 0$ we always have $\eta_k \leq 1$, provided that $\lambda \geq 0$. Accordingly, algorithm (2.2) is stable.

3. The "Predictor-Corrector" Method for Solving Unsteady Problems

We consider the equation

$$\frac{\partial \varphi}{\partial t} + \Lambda \varphi = f.$$

The entire interval $0 \leq t \leq T$ is subdivided into partial intervals $\Delta t = \tau$ and the solution of the problem at each of these intervals is to be found in two stages. First we find an approximate solution of the problem at half the interval using the scheme /72

$$\prod_{\alpha=1}^{n}\left(E + \frac{\tau}{2} A_\alpha\right)\varphi^{j+1/2} = \varphi^{j} + \frac{\tau}{2} f^{j+1/2}. \tag{3.1}$$

After the solution of this problem is obtained, we shall seek the solution for φ^{j+1} using the corrector

$$\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + \Lambda \varphi^{j+1/2} = f^{j+1/2}. \tag{3.2}$$

Here, as before,

$$\sum_{\alpha=1}^{n} A_\alpha = \Lambda + O(\tau).$$

In the simplest case, when $A_\alpha = \Lambda_\alpha$, we get

$$\left.\begin{aligned} &\prod_{\alpha=1}^{n}\left(E + \frac{\tau}{2} \Lambda_\alpha\right)\varphi^{j+1/2} = \varphi + \frac{\tau}{2} f^{j+1/2}; \\ &\frac{\varphi^{j+1} - \varphi^{j}}{\tau} + \Lambda \varphi^{j+1/2} = f^{j+1/2}. \end{aligned}\right\} \tag{3.3}$$

We now show that scheme (3.3) is of the second order of accuracy with respect to τ. In fact, multiplying the second of equations (3.3) by Λ^{-1}, and then by $\prod_{\alpha=1}^{n}\left(E + \frac{\tau}{2}\Lambda_\alpha\right)$, we get

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\Lambda^{-1}\frac{\varphi^{j+1}-\varphi^j}{\tau}+\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\varphi^{j+1/2}=$$

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\Lambda^{-1}f^{j+1/2}.$$

Then the equation thus obtained is multiplied by Λ and

$$\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\varphi^{j+1/2}$$

is eliminated using the first of equations (1.5). This yields

$$\Lambda\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\Lambda^{-1}\frac{\varphi^{j+1}-\varphi^j}{\tau}+\Lambda\varphi+\frac{\tau}{2}\Lambda f^{j+1/2}=\Lambda\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\Lambda^{-1}f^{j+1/2}$$

or

/73

$$B\frac{\varphi^{j+1}-\varphi^j}{\tau}+\Lambda\varphi^j=f^{j+1/2}+\frac{\tau^2}{4}\varepsilon^{j+1/2},$$

where

$$B=\Lambda\prod_{\alpha=1}^{n}\left(E+\frac{\tau}{2}\Lambda_\alpha\right)\Lambda^{-1};\quad \varepsilon^{j+1/2}=\Lambda R\Lambda^{-1}f^{j+1/2}.$$

The assertion that the predictor-corrector scheme (3.3) has the second order of accuracy on smooth solutions is thus proved.

Equations (3.1) and (3.2) are implemented according to the scheme:

$$\left(E+\frac{\tau}{2}\Lambda_1\right)\varphi^{j+\frac{1}{2n}}=\varphi^j+\frac{\tau}{2}f^{j+1/2};$$

$$\left(E+\frac{\tau}{2}\Lambda_2\right)\varphi^{j+\frac{2}{2n}}=\varphi^{j+\frac{1}{2n}};$$

. . . . . .

$$\left(E+\frac{\tau}{2}\Lambda_n\right)\varphi^{j+\frac{1}{2n}}=\varphi^{j+\frac{n-1}{2n}};$$

$$\varphi^{j+1}=\varphi^j-\tau(\Lambda\varphi^{j+1/2}-f^{j+1/2}).$$

It is easy to see that the predictor-corrector scheme is not equivalent to the universal algorithm scheme, but is close to it.

In form (3.3) the predictor-corrector method is satisfactorily integrated. In fact, the first equation provides for a large stability margin, which is then lost in accuracy correction. It should be noted that the predictor-corrector method satisifes the conservation laws (the system is divergent), while other methods may be also nondivergent.

The predictor-corrector method can be used also for solving steady problems in the following form:

$$\left(E + \frac{\tau_1}{2} A_1\right) \varphi^{j+\frac{1}{2n}} = \varphi^{j} + \frac{\tau}{2} f;$$

$$\left(E + \frac{\tau_2}{2} A_2\right) \varphi^{j+\frac{2}{2n}} = \varphi^{j+\frac{1}{2n}};$$

. . . . . .

$$\left(E + \frac{\tau_n}{2} A_n\right) \varphi^{j+\frac{1}{2}} = \varphi^{j+\frac{n-1}{2n}};$$

$$\varphi^{j+1} = \varphi^{j} - \tau(\Lambda\varphi^{j+\frac{1}{2}} - f).$$

/7

Here we again have n operators A_α and n + 1 parameters $\tau_1, \tau_2, \ldots, \tau_n, \tau$, which are selected from the condition of process optimization.

4. Operators With Arbitrary Structure

In solving equations of mechanics of continuous media one almost always has to deal with operators which have imaginary values as spectral points, which makes the application of general decomposition methods impossible. This corresponds to the wave nature of the solutions. It is true that equations of hyperbolic, elliptical or parabolic type serve as individual elements of the algorithm, and methods for decomposing these types of equations have already been extensively studied by Yanenko, Samarskiy, D'yakonov, and others. In addition, problems of mechanics of continuous media frequently do not belong to the class of Cauchy-Kovalevskaya problems.

As an illustration we consider the solution of the Cauchy problem

$$\frac{d\varphi}{dt} = 0$$

or

$$\frac{\partial \varphi}{\partial t} + u_1 \frac{\partial \varphi}{\partial x} + u_2 \frac{\partial \varphi}{\partial y} + u_3 \frac{\partial \varphi}{\partial z} = 0$$

with applicable initial

$$\varphi(x, y, z, 0) = \varphi_0(x, y, z)$$

and boundary conditions. We shall seek an approximate solution using difference schemes satisfying the following requirements: 1) second order of accuracy with respect to all the variables; 2) simplicity of implementation; 3) divergence.

We shall show that for given μ_i = const one such scheme will be as follows:

$$(E + \mu_i \nabla_1)(E + \mu_2 \nabla_2)(E + \mu_3 \nabla_3)(\varphi^{j+1} - \varphi^j) + (\mu_1 \Delta_1 + \mu_2 \Delta_2 + \mu_3 \Delta_3)\varphi = 0, \tag{4.1}$$

which is implemented according to the scheme /75

$$(\mu_1 \nabla_1 + E)\varphi^{j+1/3} = \varphi^j;$$

$$(\mu_2 \nabla_2 + E)\varphi^{j+2/3} = \varphi^{j+1/3};$$

$$(\mu_3 \nabla_3 + E)\varphi^{j+1} = \varphi^{j+2/3};$$

$$\varphi^{j+1} = \varphi^j - (\mu_1 \Delta_1 + \mu_2 \Delta_2 + \mu_3 \Delta_3)\varphi^{j+1}.$$

where

$$(\nabla \xi)_k = \xi_k - \xi_{k-1}; \quad (\Delta \xi)_k = \xi_{k+1} - \xi_{k-1}; \quad \mu_i = \frac{u_i \Delta t}{\Delta x_i}.$$

Let

$$\xi^j_{k_1 k_2 k_3} = \lambda^j e^{ik_1\alpha_1 + ik_2\alpha_2 + ik_3\alpha_3}. \tag{4.2}$$

Substitution of Eq. (4.2) into Eq. (4.1) yields

$$\lambda = \frac{\mathrm{Re}\, D + i\{\mathrm{Im}\, D - 2(y_1 + y_2 + y_3)\}}{\mathrm{Re}\, D + i\, \mathrm{Im}\, D}, \tag{4.3}$$

where

$$D = (1 + x_1 + iy_1)(1 + x_2 + iy_2)(1 + x_3 + iy_3);$$

$$x_k = 2\mu_k \sin^2 \frac{\alpha_k}{2}; \quad y_k = \mu_k \sin \alpha_k \qquad (k = 1, 2, 3).$$

We calculate

$$|\lambda| = \sqrt{\frac{\mathrm{Re}^2 D + \mathrm{Im}^2 D - 4\Phi}{\mathrm{Re}^2 D + \mathrm{Im}^2 D}},$$

where

$$\Phi = (y_1 + y_2 + y_3)[y_1(x_2 + x_3 + x_2 x_3) + y_2(x_1 + x_3 + x_1 x_3) + y_3(x_1 + x_2 + x_1 x_2) - y_1 y_2 y_3].$$

We now find the minimum of function $\Phi(x_1, x_2, x_3; y_1, y_2, y_3)$. Since function Φ is symmetrical relative to x_1, x_2, x_3 and y_1, y_2, y_3, then an absolute extremum of the function is possible only when

$$x_1 = x_2 = x_3 = x; \quad y_1 = y_2 = y_3 = y.$$

which yields

$$\Phi(x, y) = 3y^2 x[2 + x - y^2].$$

From which, using Eq. (4.3), we get /76

$$\mu < 3$$

as the stability region. In the case of two variables (x, y) scheme (4.1) has absolute stability for any μ.

Now it is necessary to solve the problem

$$(E + \mu \nabla)\xi = f, \tag{4.4}$$

where μ may change sign. Equation (4.4) are written in the form

$$\varphi_k + \mu_k \begin{cases} \xi_k - \xi_{k-1} & \text{for } \mu_k > 0 \\ \xi_{k+1} - \xi_k & \text{for } \mu_k < 0 \end{cases} = f_k. \tag{4.5}$$

which is then rewritten as

$$(|\mu_k| + \mu_k)\xi_{k-1} - 2(1 + |\mu_k|)\xi_k + (|\mu_k| - \mu_k)\xi_{k+1} = 2f_k \qquad (4.6)$$

together with the boundary conditions

$$\xi_0 = \alpha; \quad \xi_N = \beta.$$

Equation (4.6) is solved by the sweep method. It can be shown that computations using sweep formulas will be stable.

5. Solving Problems of Dynamic Meteorology

The main problem of dynamic meteorology is the system of equations

$$\left.\begin{aligned} &\frac{du}{dt} - lv = -RT\frac{\partial \varphi}{\partial x}; \quad \frac{d\vartheta}{dt} + \frac{\gamma_a - \gamma}{T} w = \frac{\varepsilon}{c_p}; \\ &\frac{dv}{dt} + lu = -RT\frac{\partial \varphi}{\partial y}; \\ &\frac{dw}{dt} - l\vartheta = -RT\frac{\partial \varphi}{\partial z}; \quad \operatorname{div}\vec{u} = \frac{g - \gamma R}{RT} w. \end{aligned}\right\} \qquad (5.1)$$

which should be supplemented by appropriate boundary conditions.

How can this complicated system of equations be simplified using the previously discussed ideas?

Frequently one has to deal with extensively studied problems of parabolic, elliptical or hyperbolic type. In consideration of our problem the mathematical aspect ties in quite satisfactorily with the physical aspect. We shall attempt to show how this system should be decomposed. For this one should imagine a physical process which consists in the following: there exists, let us say, a volume element of a liquid or a gas, which is to be tracked. This volume element in general moves along a specific path, but during its entire motion it adapts itself to laws of conservation of energy, momentum and mass. If a particle undergoes an infinitesimal displacement along the path then dynamic mismatch of the fields occurs. This mismatch is eliminated by the fact that, setting into operation the conservation laws, we shall attempt to distribute all the discrepancies in a special way over the field and then again to match the latter at some stage, using wave disturbances (which are permitted by our system of equations). After the fields are rematched and corrected values of fields of hydrodynamic elements are obtained, we can again move along the path and again match the fields. /77

We thus have two effects which must be considered: transfer of the substance along a path (which is the more substantial factor) and dynamic matching. Had there additionally existed eddy exchange, we would add here a third step: first our

particles would be moved along their path, then eddy exchange would be "switched on" and this will be followed by dynamic matching. Once this is understood, it becomes clear how the starting system is to be decomposed. We write the two systems into which it is decomposed. Let $\Delta t = \tau$ be a time interval from time t_j to t_{j+1}. Starting with time t_j we begin moving the particles along their path, i.e., solving the system

$$\left.\begin{aligned} &\frac{du^{(1)}}{dt} = 0; \quad u^{(1)} = u^{(j)}; \\ &\frac{dv^{(1)}}{dt} = 0; \quad v^{(1)} = v^{j}; \\ &\frac{dw^{(1)}}{dt} = 0; \quad w^{(1)} = w^{j}; \\ &\frac{d\vartheta^{(1)}}{dt} = 0; \quad \vartheta^{(1)} = \vartheta^{j}; \end{aligned}\right\} \quad \text{for} \quad t = t^{j}. \tag{5.2}$$

Upon finding the solution of the above expression at time t_{j+1}, we solve the problem of dynamic matching of the fields:

$$\left.\begin{aligned} &\frac{du^{(2)}}{dt} - lv^{(2)} = -RT\frac{\partial\varphi^{(2)}}{\partial x}; \quad u^{(2)}(t_j) = u^{(1)}(t_{j+1}); \\ &\frac{dv^{(2)}}{dt} + lu^{(2)} = -RT\frac{\partial\varphi^{(2)}}{\partial y}; \quad v^{(2)}(t_j) = v^{(1)}(t_{j+1}); \\ &\frac{dw^{(2)}}{dt} - g\vartheta^{(2)} = -RT\frac{\partial\varphi^{(2)}}{\partial z}; \quad w^{(2)}(t_j) = w^{(1)}(t_{j+1}); \\ &\frac{d\vartheta^{(2)}}{dt} + \frac{\gamma_\alpha - \gamma}{T}w^{(2)} = \frac{\varepsilon}{c_p}; \quad \vartheta^{(2)}(t_j) = \vartheta^{(1)}(t_{j+1}); \\ &\operatorname{div}\vec{u}^{(2)} = \frac{g - \gamma R}{RT}w^{(2)}. \end{aligned}\right\} \tag{5.3}$$

/78

Then Eq. (5.1) at t_{j+1} is solved in the form

$$u^{j+1} = u^{(2)}(t_{j+1}); \quad v^{j+1} = v^{(2)}(t_{j+1}); \quad w^{j+1} = w^{(2)}(t_{j+1});$$
$$\vartheta^{j+1} = \vartheta(t_{j+1}); \quad \varphi^{j+1} = \varphi^{(2)}(t_{j+1}).$$

It is seen that the weather forecasting problem has been decomposed into two simple problems, which in their turn can be decomposed further.

The solution of Eqs. (5.2) was already discussed. Let us now examine the solution of Eqs. (5.3), for which we write them in the difference form

$$u^{j+1} - u^j = \alpha v^{j+1} - c^2 \tau \varphi_x^{j+1};$$
$$v^{j+1} - v^j = -\alpha u^{j+1} - c^2 \tau \varphi_y^{j+1};$$
$$w^{j+1} - w^j = g \tau \vartheta^{j+1} - c^2 \tau \varphi_z^{j+1};$$
$$\vartheta^{j+1} - \vartheta^j = -\frac{\gamma_0 - \gamma}{T} \tau w^{j+1};$$
$$\operatorname{div} \vec{u}^{j+1} = \sigma w^{j+1};$$

where superscripts (2) are omitted for simplicity. The notation used here is

$$c^2 = RT; \quad \alpha = e\tau; \quad \tau = \Delta t; \quad \sigma = \frac{g - \gamma R}{RT}.$$

This system yields

$$u^{j+1} = -\frac{1}{1+\alpha^2}[c^2 \tau (\varphi_x + \alpha \varphi_y)^{j+1} - (u + \alpha v)^j];$$
$$v^{j+1} = -\frac{1}{1+\alpha^2}[c^2 \tau (\varphi_y - \alpha \varphi_x)^{j+1} - (v - \alpha u)^j]; \qquad /79$$
$$w^{j+1} = -\chi (c^2 \tau \varphi_z^{j+1} - g \tau \vartheta^j - w^j);$$
$$\vartheta^{j+1} = \chi \left[\vartheta^j + \frac{\gamma_\alpha - \gamma}{T} \tau (c^2 \tau \varphi_z^{j+1} - u^j)\right],$$

where $\chi = \left(1 + \frac{\gamma_\alpha - \gamma}{T} g \tau^2\right)^{-1}$ is a dimensionless parameter. Eliminating all the functions with the exception of φ and using the equation

$$\operatorname{div} \vec{u}^{j+1} = \sigma \omega^{j+1},$$

we arrive at an expression for φ^{j+1}

$$\left[\frac{\partial^2}{\partial z^2} - \delta \frac{\partial}{\partial z} + \frac{c^2 \tau}{1+\alpha^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \beta \frac{\partial}{\partial x}\right)\right] \varphi^{j+1} = -f^j, \qquad (5.4)$$

where

$$\delta = \text{const}, \quad f^j = F(u^j, v^j, w^j, \vartheta^j).$$

Equation (5.4) can be decomposed further.

6. Methods of Steadying

We shall prove, without reference to the hydrodynamic statement, a number of assertions which can be used for finding a solution to the problem when the operator of the steady-state problem is not positively defined.

Let it be required to solve the problem

$$A\varphi = f,$$

where A is not a positive definite operator. Instead of this problem we consider the unsteady problem

$$\frac{\partial \psi}{\partial t} + A\psi = f. \tag{6.1}$$

It is now asserted that the solution of Eq. (6.1) is the limit of some function

$$\varphi_T = \frac{1}{T}\int_0^T \psi\, dt \quad \text{as} \quad T \to \infty.$$

We shall prove this on the assumption that the solution of Eq. (6.1) is a bounded function. It is known that, if all the roots of an equation are imaginary, the solution is always bounded; if however, there exists a negative root $\lambda < 0$, this will result in a solution increasing exponentially with time. This will be discussed below. Applying the operator $\frac{1}{T}\int_0^T dt$ to Eq. (6.1), we have /80

$$\frac{1}{T}\int_0^T \left[\frac{\partial \psi}{\partial t} + A\psi - f\right] dt = 0.$$

Integration yields

$$\frac{\psi(T) - \psi(0)}{T} + A\varphi_T = f.$$

If solution ψ of unsteady problem (6.1) belongs to the class of bounded functions, then it is possible to find a T such that starting with it

$$\left|\frac{\psi(T)-\psi(0)}{T}\right| < \varepsilon,$$

and, in fact, we get uniform convergence, irrespective of x, y, z.

The solution of unsteady problems is obtained with sufficiently high accuracy, since one can use for them the universal algorithm and reduce the solution of this problem to sequential solution of elementary one-dimensional problems, which was discussed previously.

We considered a quite simple problem of steadying and found in this case that its convergence to the solution of the steady-state problem was of order 1/T. We now wish to know whether it is possible to construct methods so that, having quite arbitrary solutions and applying to them some more complex operators, the convergence to the steady problem is of order $1/T^n$.

We consider one particular case. Let it be required to solve the problem

$$\Lambda\varphi = f.$$

Instead of it we shall consider

$$\frac{\partial^n \psi}{\partial t^n} + \Lambda\psi = f. \qquad (6.2)$$

Since t is a formal variable, in solving this problem we select the initial conditions as follows:

/81

$$\psi(0) = \psi_t'(0) = \ldots = \psi_t^{(n-1)}(0) = 0.$$

If the solution of Eq. (6.2) is bounded, then

$$\varphi = \lim_{T\to\infty} \varphi_T = \lim_{T\to\infty} \frac{n!}{T^n} \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} \psi\, dt_n,$$

and in the uniform metric

$$\| \varphi - \varphi_T \| = 0\left(\frac{1}{T^n}\right).$$

In fact, let us apply operator

$$\frac{n!}{T^n}\int_0^T dt_1\int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

to Eq. (6.2). As a result, with consideration of initial conditions, we get

$$\frac{\psi(T)}{T^n} + \Lambda\varphi_T = f \qquad \text{or} \qquad \Lambda\varphi_T = f + 0\left(\frac{1}{T^n}\right),$$

since $\psi(T)$ belongs to the class of bounded functions.

This means that if the problem converges slowly, other steadying operators should be used. This increases the order of equations for t.

Up to now we have considered methods for solving equations such as (6.1) by the decomposition method. In some cases the decomposition of wave equations has been considered (Samarskiy, D'yakonov, Konovalov). Wave equations correspond to the case when Λ is an elliptical Laplace operator and n = 2. In the particular case of a wave equation, when instead of problem

$$\Lambda\varphi = f$$

we consider

$$\frac{\partial^2\psi}{\partial t^2} + \Lambda\psi = f; \quad \psi(0) = \psi'(0) = 0,$$

we have

$$\frac{\psi(T)}{T^2} + \Lambda\varphi_T = f.$$

Consequently, for function

$$\varphi_T = \frac{2}{T^2}\int_0^T dt_1\int_0^{t_1}\psi\, dt$$

the convergence is of order $1/T^2$. Steadying of such an equation was considered first by Saul'yev, who obtained convergence of order 1/T. /82

Problems with positive definite operators are the simplest. If the operator's spectrum is imaginary, then the problems become more complicated, but here one

may use all the methods utilized for positively defined operators. If, however, the operator's spectrum contains one or several negative numbers, this results in the appearance of harmonics increasing with time. Here it is no longer possible to steady the problem in principle, even when using the previously described functionals, since it is assumed everywhere that the solutions are bounded. In this case it is necessary to know how to isolate these harmonics. If this could be done, then the separation theory could be used for solving a wider class of problems.

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THE USE OF THE METHOD OF THE SMALL PARAMETER FOR NUMERICAL SOLUTION OF EQUATIONS OF MATHEMATICAL PHYSICS

A. A. Dorodnitsyn

1. The advent of electronic computers has increased by a factor of hundreds of /8
thousands and now of tens of millions the computational capabilities of mankind. This has made a substantial change in the approach to solving problems of mathematical physics. While during the "pre-computer" era investigators attempted to move as far as possible in analytic solution of the problem, thus leaving a minimum of computational work, the appearance of high-speed electronic computers resulted in an opposite trend - to "arithmetize" the problem as fast as possible, i.e., to write it (as a rule, approximately) in a form in which the subsequent solution would be reduced to arithmetic operations, i.e., could be carried out by computer.

As a result the powerful analytic tools which were developed during the pre-computer era were to some extent temporarily put aside, if not forgotten.

However, in spite of the high speed at which computer technology is being refined, the complexity of problems generated by modern physics and technology increases faster than the capability of computers, and the direct "arithmetization" approach frequently results in computations so complex that they are beyond the capability of even the most modern machines. It is therefore natural to turn to the store of knowledge accumulated by "analytic" mathematics and to see whether one can find there effective methods which, together with machine computations, would make possible easier solution of complex problems of mathematical physics.

The small-parameter method has been quite effective in many analytical studies. This method substantially (although not explicitly) underlies all the /8
methods of successive approximation, but is most clearly expressed in perturbation theory.

As to the "small parameter" proper, in some problems it has a direct physical meaning, while in others it is introduced artificially as a certain formal parameter. In the latter case the starting, specified problem is obtained for some specific value (usually equal to unity, i.e., not small) value of the parameter.

2. The introduction to the small-parameter problem is a rather standard operation.

Let us have a starting equation, which is written in the general operator form

$$\Phi(\vec{x}, \vec{u}) = 0, \qquad (2.1)$$

where $\vec{x}$ is the radius vector of points in the domain where the solution is sought, and $\vec{u}$ is the sought vector function (the dimensions of $\vec{x}$ and $\vec{u}$, obviously, can be entirely different).

From a certain set of considerations we find the equation

$$\Phi_0(\vec{x}, \vec{u}) = 0,$$

the solution of which, we hope, is close to the solution of the starting equation. Writing the starting equation in the form

$$\Phi(\vec{x}, \vec{u}) = \Phi_0(\vec{x}, \vec{u}) + \varphi(\vec{x}, \vec{u}) = 0;$$

where

$$\varphi(\vec{x}, \vec{u}) = \Phi(\vec{x}, \vec{u}) - \Phi_0(\vec{x}, \vec{u}),$$

we construct the more general equation

$$\Phi_0(\vec{x}, \vec{u}) + \varepsilon\varphi(\vec{x}, \vec{u}) = 0 \qquad (2.2)$$

and solve it by one method or another. Solution $\vec{u}$ of this latter equation is a function of ε

$$\vec{u} = \vec{u}(\vec{x}, \varepsilon),$$

and the solution of starting equation (2.1) is obtained with $\varepsilon = 1$. The introduction of the small parameter ε did not simplify the problem, but now we are able to study the solution of Eq. (2.2) as a function of this parameter. The idea of introducing the parameter consists in the fact that it may be simpler to clarify the dependence of the solution on ε, then to immediately find $\vec{u}$ as a function of $\vec{x}$ in the starting equation. /87

It is usually not difficult to show that $\vec{u}(\vec{x}, \varepsilon)$ is a continuous, differentiable (and even analytical in some range of values of ε) function of ε. This makes it possible to consider a derivative of $\vec{u}$ with respect to ε:

$$v(\vec{x}, \varepsilon) = \frac{\partial \vec{u}}{\partial \varepsilon}.$$

The equation will be linear in $\vec{v}$

$$\left(\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}\right)\vec{v} + \varphi(\vec{x}, \vec{u}) = 0.$$

If the operator which is the inverse of operator

$$\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}$$

(we denote it by $\left[\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}\right]^{-1}$) is easily arithmetized, then the problem can be arithmetized easily by using a method, let us say, similar to the Euler method in ordinary differential equations. We have

$$\vec{v}(\vec{x}, \varepsilon) = -\left[\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}\right]^{-1} \varphi(\vec{x}, \vec{u});$$

$$\vec{u}(\vec{x}, \varepsilon + \Delta\varepsilon) = \vec{u}(\vec{x}, \varepsilon) + \Delta\varepsilon \cdot \vec{v}(\vec{x}, \varepsilon)$$

and $\vec{u}$ is thus found step by step up to the value of $\varepsilon = 1$. Naturally, this can be done in the case when $\vec{u}$ as a function of ε is a continuous differentiable function in the interval $0 \leq \varepsilon \leq 1$.

However, in general, the finding of operator $\left[\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}\right]^{-1}$ is not a simple task, although for a sufficiently good direct operator it can be approximately reduced to an algebraic linear system of equations. But, since the direct operator $\frac{\partial \Phi_0}{\partial \vec{u}} + \varepsilon \frac{\partial \varphi}{\partial \vec{u}}$ is a function of ε and $\vec{u}$, in general this linear system has to be solved again at each step in ε. The method will be effective when the inverse operator is easily calculated, but this is not a frequent occurrence. /88

It is much simpler to subdivide the starting direct operator into the sum $\Phi_0 + \varphi$ in such a manner that $\left[\frac{\partial \Phi_0}{\partial \vec{u}}\right]^{-1}$ can be easily found. For example, if $\Phi_0(\vec{x}, \vec{u})$ is linear in $\vec{u}$, then operator $\left[\frac{\partial \Phi_0}{\partial \vec{u}}\right]^{-1}$ does not depend on $\vec{u}$ and ε, and can be precalculated.

If, as in the ordinary perturbation theory, the solution is sought as a power series in ε, then to calculate each coefficient one needs precisely only one operator

$$\left[\frac{\partial \Phi_0}{\partial \vec{u}}\bigg|_{\varepsilon=0}\right]^{-1},$$

since, if we set

$$\vec{u}(\vec{x}, \varepsilon) = \sum_0^\infty \varepsilon^n \vec{v}_n(\vec{x}),$$

we get for $v_n(\vec{x})$ the expression

$$\frac{\partial \Phi_0}{\partial \vec{u}}\bigg|_{\vec{u}=v_0} \vec{v}_n = F(\vec{x}, \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1}). \tag{2.3}$$

For computational purposes a series expansion in ε is inconvenient for two reasons. The right-hand sides of Eq. (2.3) for any complex nonlinear equations grow very rapidly, and this requires complex analytical transformations before making the expansion. When a large number of terms of a series is to be calculated, the machine memory will be unduly loaded. Here it is more expedient to use the method of successive approximations

$$\Phi_0(\vec{x}, \vec{u}_n) + \varepsilon \varphi(\vec{x}, \vec{u}_{n-1}) = 0,$$

the effectiveness of which will depend on the simplicity of solving equation

$$\Phi_0(\vec{x}, \vec{u}) = f(\vec{x}).$$

3. The method of power series expansion in ε has the substantial shortcoming that the convergence of the series is determined by the analyticity of the solution /89
in the range $0 \leq \varepsilon \leq 1$ of interest, but depends on the behavior of the function in the circle $|\varepsilon| \leq 1$. Thus, if function $\vec{u}(\vec{x}, \varepsilon)$ has a singularity for some complex or negative ε whose absolute value is smaller than unity, then the series will diverge for $\varepsilon = 1$.

The situation may be somewhat better for the method of successive approximations, but this method also may diverge even if the function remains analytical in the interval $0 \leq \varepsilon \leq 1$. Hence it becomes attractive to employ analytical continuation with respect to ε, which would permit circumvention of the "parasitic" singular points.

The simplest approach in this direction is writing Eq. (2.1) in the form

$$\Phi_0(\vec{x}, \vec{u}) + \varepsilon_1 \{\alpha \Phi(\vec{x}, \vec{u}) - \Phi_0(\vec{x}, \vec{u})\} = 0, \tag{3.1}$$

where α is some constant. Obviously, when $\varepsilon_1 = 1$ Eq. (3.1) coincides with Eq. (2.1).

Let us find the relationship between parameters ε_1 and ε (Eq. (2.2)). Since $\Phi(\vec{x}, \vec{u}) = \Phi_0(\vec{x}, \vec{u}) + \varphi(\vec{x}, \vec{u})$, Eq. (3.1) can be rewritten in the form

$$\Phi_0 + \varepsilon_1\{\alpha(\Phi_0 + \varphi) - \Phi_0\} = [1 - \varepsilon_1(1-\alpha)]\Phi_0 + \varepsilon_1\alpha\varphi = 0$$

or

$$\Phi_0(\vec{x}, \vec{u}) + \frac{\varepsilon_1\alpha}{1-\varepsilon_1(1-\alpha)} \cdot \varphi(\vec{x}, \vec{u}) = 0. \tag{3.2}$$

Equating (3.2) and (2.2), we get

$$\varepsilon = \frac{\alpha\varepsilon_1}{1-\varepsilon_1(1-\alpha)}. \tag{3.3}$$

Function (3.3) maps a unit radius circle in the ε_1 plane into a circle with radius $1/(2 - \alpha)$ with center at point $x = \frac{1}{2} \cdot \frac{1-\alpha}{1-\frac{1}{2}\alpha}$, $y = 0$ (Fig. 1). Thus, if the solution of Eq. (2.2) has no singularities inside a circle whose center is at point $(1/2, 0)$ and whose radius is greater than $1/2 + \delta$, where δ is a positive integer as small as desired, then for a sufficiently small $\alpha > 0$ the solution of Eq. (3.1) will be analytical in ε_1 when $\varepsilon_1 \leq 1$. /90

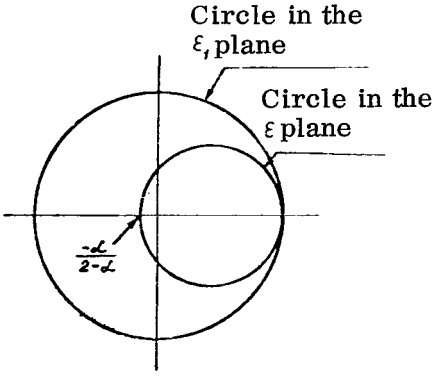

Figure 1.

A more general case of analytical extension can be obtained by replacing the constant α with one which is a function of ε_1. Then

$$\varepsilon = \frac{\alpha(\varepsilon_1)\varepsilon_1}{1-\varepsilon_1[1-\alpha(\varepsilon_1)]}.$$

It is always possible to find a function $f(\varepsilon_1)$ which maps a unit-radius circle of the ε_1 plane into as narrow a region as desired, which surrounds segment $0 \leq \varepsilon \leq 1$, and in addition such that the 0 of plane ε_1 is mapped into the 0 of the ε plane and $\varepsilon_1 = 1$ is mapped into $\varepsilon = 1$ (Fig. 2).

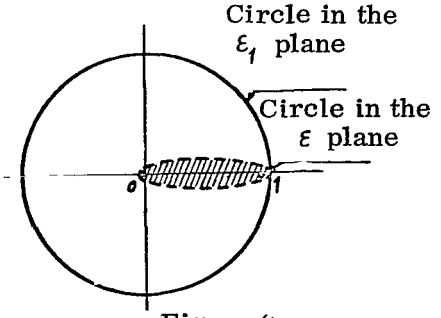

Figure 2.

Upon finding $\alpha(\varepsilon_1)$ from

$$f(\varepsilon_1) = \frac{\varepsilon_1 \alpha(\varepsilon_1)}{1 - \varepsilon_1[1 - \alpha(\varepsilon_1)]},$$

i.e.,

$$\alpha(\varepsilon_1) = \frac{f(\varepsilon_1)}{\varepsilon_1} \cdot \frac{1 - \varepsilon_1}{1 - f(\varepsilon_1)},$$

we get the following result. It is always possible to find the solution of equation

$$\Phi_0(\vec{x}, \vec{u}) + \varepsilon_1\{\alpha(\varepsilon_1)\Phi(\vec{x}, \vec{u}) - \Phi_0(\vec{x}, \vec{u})\} = 0$$

as a power series in ε_1 if the solution of Eq. (2.2) is analytical in ε in the entire range $0 \leq \varepsilon \leq 1$.

Finally we note that in some cases it is more expedient to specify an α which is a function of $\vec{x}$. This makes it possible to avoid undue difficulties in the vicinity of singular points of Eq. (2.1). However, this is not considered in detail here, but instead we examine an example.

4. Application of the above method is now described by several examples. These examples are naturally illustrational, i.e., they are sufficiently simple and easily understood.

A. We consider the classical example of a nonlinear problem for the Laplace equation — the problem of conformal mapping of some region into a unit-radius circle.

In the plane $z = re^{i\theta}$ let there be given a domain D whose boundary is specified as /91

$$\ln r = u(\theta).$$

We assume that $u(\theta)$ is a single-value differentiable function. Let the coordinate origin of plane z lie inside D. We map D into a unit circle of the plane $\zeta = \rho e^{iv}$ in such manner that the coordinate origin z = 0 is transformed to the center of the circle $\zeta = 0$, and we assume for definiteness that $(dz/d\zeta)_{\zeta=0}$ is real and positive.

We shall solve the more general problem for the domain $D(\varepsilon)$ whose boundary is given by

$$\ln r = \varepsilon u(\theta). \qquad (4.1)$$

The function mapping $D(\varepsilon)$ into a unit - radius circle depends on ζ and ε:

$$z = z(\zeta, \varepsilon),$$

wherein, for $\varepsilon = 0$,

$$z = \zeta.$$

The function

$$\omega(\zeta, \varepsilon) = \ln r + i\theta = \tau + i\theta$$

is an analytic function of ζ in the circle $|\zeta| < 1$ with a singular (logarithmic) point at the coordinate origin, but $\omega(\zeta, \varepsilon) - \ln\zeta$ is analytical in the entire circle. Hence, also the function

$$\dot{\omega}(\zeta, \varepsilon) = \frac{\partial \omega}{\partial \varepsilon}$$

is analytical inside circle $|z| < 1$. Boundary condition (4.1) may be written in the more complete form as

$$\tau(v, \varepsilon) = \varepsilon u[\theta(v, \varepsilon)] \quad \text{for} \quad \rho = 1. \tag{4.2}$$

whence

$$\frac{\partial \tau}{\partial \varepsilon} = \varepsilon u'[\theta(v, \varepsilon)] \cdot \frac{\partial \theta}{\partial \varepsilon} + u[\theta(v, \varepsilon)]$$

or, using the notation

$$\frac{\partial \tau}{\partial \varepsilon} = \dot{\tau}; \quad \frac{\partial \theta}{\partial \varepsilon} = \dot{\theta},$$

we get

$$\dot{\tau}(v, \varepsilon) = \varepsilon u'[\theta(v, \varepsilon)]\,\dot{\theta}(v, \varepsilon) + u[\theta(v, \varepsilon)]. \tag{4.3}$$

Equation (4.3) is already a linear condition for the boundary values of function $\dot{\omega}(\zeta, \varepsilon)$. However, the coefficients in this condition themselves are functions of ε, and solving this problem, let us say, by the Euler method, we would have been forced to seek a solution of a boundary value problem with different coefficients at each step in ε. To avoid this, we do the following. Equation (4.2) is differentiated with respect to v: /92

$$\frac{\partial \tau}{\partial v} = \varepsilon u'[\theta(v, \varepsilon)]\frac{\partial \theta}{\partial v},$$

i.e.,

$$\varepsilon u'[\theta(v, \varepsilon)] = \frac{\partial \tau}{\partial v} \Big/ \frac{\partial \theta}{\partial v},$$

and condition (4.3) can be rewritten in the form

$$\dot{\tau}(v, \varepsilon) = u[\theta(v, \varepsilon)] + \frac{\dot{\theta}(v, \varepsilon)\frac{\partial \tau}{\partial v}}{\frac{\partial \theta}{\partial v}}$$

or

$$\frac{\dot{\tau}\frac{\partial \theta}{\partial v} - \dot{\theta}\frac{\partial \tau}{\partial v}}{\left(\frac{\partial \tau}{\partial v}\right)^2 + \left(\frac{\partial \theta}{\partial v}\right)^2} = \frac{u[\theta(v, \varepsilon)]}{\left(\frac{\partial \tau}{\partial v}\right)^2 + \left(\frac{\partial \theta}{\partial v}\right)^2} \cdot \frac{\partial \theta}{\partial v}. \tag{4.4}$$

Noting that

$$i\frac{\dot{\tau} + i\dot{\theta}}{\frac{\partial \tau}{\partial v} + i\frac{\partial \theta}{\partial v}} = \frac{\left(\dot{\tau}\frac{\partial \theta}{\partial v} - \dot{\theta}\frac{\partial \tau}{\partial v}\right) + i\left(\dot{\tau}\frac{\partial \tau}{\partial v} + \dot{\theta}\frac{\partial \theta}{\partial v}\right)}{\left(\frac{\partial \tau}{\partial v}\right)^2 + \left(\frac{\partial \theta}{\partial v}\right)^2}$$

and also

$$\left.\frac{d \ln z}{d\zeta}\right|_{\rho = 1} = \frac{1}{ie^{iv}}\left(\frac{\partial \tau}{\partial v} + i\frac{\partial \theta}{\partial v}\right) = \frac{1}{i\zeta}\left(\frac{\partial \tau}{\partial v} + i\frac{\partial \theta}{\partial v}\right),$$

we have

$$i\frac{\dot{\tau} + i\dot{\theta}}{\frac{\partial \tau}{\partial v} + i\frac{\partial \theta}{\partial v}} = \frac{\dot{\omega}}{\zeta\frac{d \ln z}{d\zeta}} \quad \text{for} \quad |\zeta| = 1.$$

and condition (4.4) can thus be written in the form

$$\operatorname{Re}\left[\frac{\dot{\omega}}{\zeta\dfrac{d\ln z}{d\zeta}}\right] = \frac{u[\theta(v,\varepsilon)]}{\left(\dfrac{\partial\tau}{\partial v}\right)^2+\left(\dfrac{\partial\theta}{\partial v}\right)^2}\cdot\frac{\partial\theta}{\partial v}.$$

We have thus obtained for the function /93

$$\frac{\dot{\omega}}{\zeta\dfrac{d\ln z}{d\zeta}},$$

which is analytic inside circle $|\zeta|<1$, the classical Dirichlet problem. This function is found explicity in terms of the Poisson integral. We thus always carry out transformation from ε to $\varepsilon+\Delta\varepsilon$ by the same computational procedure, which reduces, when using finite differences, to multiplying a matrix (which is always the same for any $u(\theta)$) by a vector.

B. Let it be required to find the solution of the equation

$$\Delta\varphi - c(x,y)\varphi = f(x,y), \quad c(x,y)>0 \tag{4.5}$$

in a unit-radius circle with boundary conditions $\varphi = 0$ at the circumference $x^2 + y^2 = 1$. Writing the equation in the more general form

$$\Delta\varphi - \varepsilon c(x,y)\varphi = f(x,y),$$

we get the usual method of perturbation theory.

All the singular (with respect to ε) points of the solution of this equation are eigenvalues of the equation

$$\left.\begin{aligned}&\Delta\varphi - \lambda c(x,y)\varphi = 0,\\ &\varphi = 0 \quad \text{for} \quad x^2+y^2=1.\end{aligned}\right\} \tag{4.6}$$

All the eigenvalues are negative. Thus, the "success" in using the ordinary perturbation theory depends on whether the absolute value of the first eigenvalue of Eq. (4.6) is smaller or greater than unity. If coefficient c(x, y) is sufficiently large, then eigenvalue λ_1 can be as small as desired, and hence perturbation theory is not suitable for solving the starting equation (4.5). But, by introducing the constant α, i.e., considering the equation

$$\Delta\varphi - f + \varepsilon_1[\alpha(\Delta\varphi - c\varphi - f) - \Delta\varphi + f] = 0,$$

the solution can be found up to $\varepsilon_1 = 1$, as long as we assume that

$$\frac{\alpha}{2-\alpha} < |\lambda_1| , \quad \alpha > 0.$$

C. We consider the biharmonic equation

$$\Delta\Delta\varphi = f(x, y)$$

with different boundary conditions in some closed, bounded region. Its solution is particularly simple with boundary conditions of the form /94

$$\varphi = 0; \quad \Delta\varphi = 0 \quad \text{on} \quad \Gamma \text{ (boundary of domain)}$$

Then the solution reduces to successive solution of the Possion equations converges for

$$\Delta\omega = f(x, y), \quad \omega = 0 \quad \text{on} \quad \Gamma;$$
$$\Delta\varphi = \omega, \quad \varphi = 0 \quad \text{on} \quad \Gamma.$$

But now the problem of a rigidly fastened plate, for which the boundary conditions have the form

$$\varphi = 0; \quad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{4.7}$$

does not decompose into two 2nd-order problems.

Taking, instead of the second condition in (4.7), the condition

$$\omega + \varepsilon\left(\alpha\frac{\partial\varphi}{\partial n} - \omega\right) = 0 \quad \text{on} \quad \Gamma,$$

we can seek a solution in the form of a series in ε:

$$\omega = \sum_0^\infty \omega_n \varepsilon^n; \quad \varphi = \sum_0^\infty \varphi_n \varepsilon^n. \tag{4.8}$$

Here, for ω_n and φ_n we obtain a recurrent system of decomposing equations

$$\Delta\omega_0 = f(x, y); \quad \Delta\varphi_0 = \omega_0;$$
$$\varphi_0 = 0; \quad \omega_0 = 0 \quad \text{on} \quad \Gamma;$$
$$\Delta\omega_k = 0; \quad \Delta\varphi_k = \omega_k;$$
$$\varphi_k = 0; \quad \omega_k = -\alpha\frac{\partial\varphi_{k-1}}{\partial n} + \omega_{k-1} \quad \text{on} \quad \Gamma.$$

As was shown by B. Pal'tsev, series (4.8) converge provided that

$$\alpha > 0; \qquad \alpha < \frac{K}{\sqrt{D}},$$

where D is the area of the domain and K is some constant depending on the domain's form (it is assumed that the domain's boundary is sufficiently smooth). In order to make this result more graphic, we consider the quite simple case of a one-dimensional "biharmonic" equation

$$\frac{d^2\omega}{dx^2} = f(x); \quad \frac{d^2\varphi}{dx^2} = \omega \qquad (0 \leq x \leq a) \tag{4.9}$$

with the boundary conditions /95

$$\left.\begin{aligned} &\varphi(0) = 0, \quad \varphi(a) = 0; \\ &\omega(0) = 0, \quad \left.\frac{\partial \varphi}{\partial x}\right|_{x=a} = 0. \end{aligned}\right\} \tag{4.10}$$

Had, for x = a, the value of ω been specified instead of the condition $\frac{\partial \varphi}{\partial x} = 0$, then the fourth-order system (4.9) would have separated into two 2nd-order systems. Under conditions (4.10) this separation does not occur and we must directly solve a fourth-order system. We now replace the last of conditions (4.10) with the more general condition

$$\text{for } x = a: \qquad \omega + \varepsilon\left(\alpha \frac{\partial \varphi}{\partial x} - \omega\right) = 0, \tag{4.11}$$

which reverts to that of (4.10) for $\varepsilon = 1$.

Since the study of system (4.9) with conditions (4.11) is elementary, we can easily observe the dependence of this solution on ε and the conditions under which the power series in ε for functions ω and φ converges for $\varepsilon = 1$.

From the first of equations (4.9) we get

$$\omega(x) = \Omega_0(x) + c_1 x,$$

where $\Omega_0(x)$ is a particular solution, let us say such that $\Omega_0(0) = 0$ and $\Omega_0'(0) = 0$ (i.e., $\Omega_0(x) = \int_0^x ds \int_0^s f(s_1)\, ds_1$). We made use here of the condition $\omega(0) = 0$. Now, from the second equation we have

$$\varphi(x) = \Phi_0(x) + c_1\frac{x^3}{6} + c_2 x,$$

where $\Phi_0(x)$ is the particular solution $\left[\Phi_0(x) = \int_0^x ds \int_0^s \Omega_0(s_1)ds_1\right]$, corresponding to the term Ω_0 in the expression for ω. Constants c_1 and c_2 will be found from condition (4.11) and the condition $\varphi(a) = 0$

$$\Phi_0(a) + c_1\frac{a^3}{6} + c_2 a = 0;$$

$$\Omega_0(a) + c_1 a + \varepsilon\left\{\alpha\left[\Phi_0'(a) + \frac{c_1 a^2}{2} + c_2\right] - \Omega_0(a) - c_1 a\right\} = 0.$$

Elementary calculations yield

$$c_1 = \frac{\varepsilon\alpha[\Phi_0(a) - \Phi_0'(a)] - a(1-\varepsilon)\Omega_0(a)}{a^2\left[1 - \varepsilon\left(1 - \frac{\alpha a}{3}\right)\right]};$$

$$c_2 = \frac{\frac{a^2}{6}(1-\varepsilon)\Omega_0(a) + \frac{a^2}{6}\varepsilon\alpha\Phi_0'(a) - \left(1 - \varepsilon - \frac{\varepsilon\alpha a}{2}\right)\Phi_0(a)}{a\left[1 - \varepsilon\left(1 - \frac{\alpha a}{3}\right)\right]}. \qquad \text{/96}$$

Thus, the solution of system of equations (4.9) can be expanded in a power series in ε provided that $\left|\varepsilon\left(1 - \frac{\alpha a}{3}\right)\right| < 1$; consequently, when

$$0 < \alpha < \frac{6}{a}$$

the series converges for $\varepsilon = 1$ and yields the solution of the starting problem. This result, as is seen, is qualitatively the same as for the general biharmonic equation. Constant K is here determined exactly (K = 6).

D. Finally we consider the more complicated case of the flow of a viscous fluid. The equation of a viscous incompressible fluid can be reduced to a single fourth-order equation

$$\Delta\Delta\psi = 2\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right). \qquad (4.12)$$

The boundary conditions at the solid walls have the form

$$\psi = \text{const}; \qquad \frac{\partial \psi}{\partial n} = 0.$$

One of the important problems in the theory of viscous fluids is the flow past a semi-infinite plate. Here the boundary conditions are written in the form

$$\psi(x, 0) = 0; \ \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (y = 0, \ x < 0); \ \frac{\partial \psi}{\partial y} = 0 \quad (y = 0, \ x > 0);$$

$$\frac{\partial \psi}{\partial y} \to 1 \quad \text{for} \quad x^2 + y^2 \to \infty \quad \left(v = \operatorname{arctg} \frac{y}{x} > \delta > 0 \right).$$

In Oseen's approximate statement, in which it is assumed that ψ differs little from values for a uniform flow ($\psi = y$), Eq. (4.12) is replaced by the expression

$$\Delta \Delta \psi = 2 \frac{\partial \Delta \psi}{\partial x}$$

with the same boundary conditions.

If we introduce the vorticity $\omega = \Delta \psi$, then Eq. (4.12) and the boundary conditions can be rewritten in the form

$$\Delta \omega = 2 \frac{\partial \omega}{\partial x}; \quad \psi = 0 \quad \text{for} \quad y = 0;$$

$$\Delta \psi = \omega; \quad \omega = 0 \quad \text{for} \quad y = 0, \ x < 0; \qquad /9$$

$$\frac{\partial \psi}{\partial y} = 0 \quad \text{for} \quad y = 0, \ x > 0.$$

These equations can be "separated" in the ordinary manner by writing condition $x > 0$ in the form

$$\omega + \varepsilon \left[-\alpha \frac{\partial \psi}{\partial y} - \omega \right] = 0 \quad \text{for} \quad y = 0, \ x > 0.$$

However, in this case the domain is infinite, function ψ is not bounded, and the fact that the asymptotic behavior of function ω at infinity is not taken into account will result in divergence of the series in ε. But the asymptotic of the solution as $x \to \infty$ is known: $\omega \sim \frac{1}{\sqrt{x}}$. Hence we write the "generalized" equation in the form

$$\omega + \varepsilon \left[-\frac{\alpha}{\sqrt{x}} \cdot \frac{\partial \psi}{\partial y} - \omega \right] = 0.$$

The expansion of functions ω and ψ in a series in ε is

$$\omega = \sum_1^\infty \omega_n \cdot \varepsilon^n; \qquad \psi = y + \sum_1^\infty \psi_n \cdot \varepsilon^n,$$

and for ω_n and ψ_n we get the recurrent formulas

$$\Delta\omega_n - 2\frac{\partial\omega_n}{\partial x} = 0; \quad \omega_n(x, 0) = 0 \qquad (x < 0);$$
$$\omega_n(x, 0) = \frac{\alpha}{\sqrt{x}}\frac{\partial\psi_{n-1}}{\partial y} + \omega_{n-1} \qquad (x > 0);$$
$$\Delta\psi_n = \omega_n; \qquad \psi_n(x, 0) = 0.$$

(ω_n and grad ψ_n tend to zero at infinity). These equations are solved simply. First we find ω_1 and ψ_1. For ω_1 we get the boundary condition

$$\omega_1(x, 0) = 0 \quad (x < 0); \qquad \omega_1(x, 0) = \frac{\alpha}{\sqrt{x}} \quad (x > 0);$$
$$\omega_1 \to 0 \quad \text{for} \quad x^2 + y^2 \to \infty.$$

It is easy to check by direct substitution that

$$\omega_1 = \alpha e^x \cdot \frac{e^{-r}}{\sqrt{r}} \cdot \cos\frac{1}{2}v$$

(r and v are polar coordinates). In order to find ψ_1 it is more convenient to convert to the parabolic coordinates

$$x = \xi^2 - \eta^2; \qquad y = 2\xi\eta.$$

in which the equation for ψ_1 is written in the form /98

$$\frac{\partial^2\psi_1}{\partial\xi^2} + \frac{\partial^2\psi_1}{\partial\eta^2} = 4(\xi^2 + \eta^2)\omega_1;$$

$$\omega_1(\xi, \eta) = \alpha\xi\frac{e^{-2\eta^2}}{\xi^2 + \eta^2}$$

with the conditions

$$\psi_1(0, \eta) = 0; \quad \psi_1(\xi, 0) = 0;$$
$$\text{grad } \psi_1 \to 0 \quad \text{for} \quad \xi^2 + \eta^2 \to \infty.$$

Setting

$$\psi_1 = \xi \Psi(\eta),$$

we get

$$\Psi'' = 4\alpha e^{-2\eta^2},$$

whence

$$\Psi = -\alpha\left[4\eta \int_{\eta}^{\infty} e^{-2s^2} ds + \left(1 - e^{-2\eta^2}\right)\right]$$

and

$$\psi_1 = -\alpha\xi\left[4\eta \int_{\eta}^{\infty} e^{-2s^2} ds + \left(1 - e^{-2\eta^2}\right)\right]$$

(all the boundary conditions are incorporated here). Let us now find $\left.\frac{\partial \psi_1}{\partial y}\right|_{y=0}$.

The conversion formulas

$$\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_1}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_1}{\partial \eta}\frac{\partial \eta}{\partial y}; \quad \frac{\partial \xi}{\partial y} = \frac{\eta}{2(\xi^2 + \eta^2)}; \quad \frac{\partial \eta}{\partial y} = \frac{\xi}{2(\xi^2 + \eta^2)}$$

yield

$$\left.\frac{\partial \psi_1}{\partial y}\right|_{y=0} = -2\alpha \int_0^{\infty} e^{-2s^2} ds = -\alpha\sqrt{\frac{\pi}{2}}.$$

We thus see that $\left.\frac{\partial \psi_1}{\partial y}\right|_{y=0}$ is a constant. Hence we get for ω_2 a boundary condition of the same form as for ω_1, but with a different multiplier

$$\text{for } y = 0, \ x > 0: \quad \omega_2 = \frac{\alpha}{\sqrt{x}}\left(-\alpha\sqrt{\frac{\pi}{2}}\right) + \frac{\alpha}{\sqrt{x}} = \frac{\alpha}{\sqrt{x}}\left(1 - \alpha\sqrt{\frac{\pi}{2}}\right),$$

whence

/99

$$\omega_2 = \alpha\left(1 - \alpha\sqrt{\frac{\pi}{2}}\right)e^x \cdot \frac{e^{-r}}{\sqrt{r}}\cos\frac{1}{2}v,$$

and further

$$\psi_2 = -\alpha\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)\xi\left[4\eta\int_\eta^\infty e^{-2s^2}ds + \left(1-e^{-2\eta^2}\right)\right];$$

$$\left.\frac{\partial\psi_2}{\partial y}\right|_{y=0} = -\alpha\sqrt{\frac{\pi}{2}}\left(1-\alpha\sqrt{\frac{\pi}{2}}\right).$$

In general,

$$\omega_n = \alpha\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)^{n-1}e^x\,\frac{e^{-r}}{\sqrt{r}}\cos\frac{1}{2}v;$$

$$\psi_n = -\alpha\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)^{n-1}\xi\left[4\eta\int_\eta^\infty e^{-2s^2}ds + \left(1-e^{-2\eta^2}\right)\right].$$

Thus, the power series in ε converges if

$$\left|\varepsilon\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)\right| < 1.$$

For convergence of the series when $\varepsilon = 1$ it suffices that

$$0 < \alpha < \frac{2\sqrt{2}}{\sqrt{\pi}}.$$

The summation of the series is also found in an elementary manner:

$$\omega = \sum_1^\infty \omega_n\varepsilon^n = \alpha e^x \cdot \frac{e^{-r}}{\sqrt{r}}\cos\frac{1}{2}v\sum_1^\infty \varepsilon^n\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)^{n-1} =$$

$$\frac{\varepsilon\alpha}{1-\varepsilon\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)}\cdot e^x\cdot\frac{e^{-r}}{\sqrt{r}}\cos\frac{1}{2}v;$$

$$\psi = \sum_1^\infty \psi_n\varepsilon^n = y - \frac{\varepsilon\alpha\xi}{1-\varepsilon\left(1-\alpha\sqrt{\frac{\pi}{2}}\right)}\cdot\left[4\eta\int_\eta^\infty e^{-2s^2}ds + \left(1-e^{-2\eta^2}\right)\right].$$

For $\varepsilon = 1$

$$\omega = \sqrt{\frac{2}{\pi}}\, e^{x} \cdot \frac{e^{-r}}{\sqrt{r}} \cos \frac{1}{2} v ,$$

i.e., the solution, naturally, does not depend on the selection of α and is the solution of the starting problem.

We note that in the above examples there exists an "optimal" value of α ($\alpha = \sqrt{\frac{2}{\pi}}$ in the problem just considered) when the series simply breaks off and consists of one nonzero term. Obviously, this cannot happen in the general case, but it may be expected that the convergence will be optimal for some value of α. /1(

We note in conclusion that the examined examples show the advantage of introducing the small parameter — that calculation of all the approximations is performed by the "standard method." In using finite differences it is necessary to solve in each approximation a system of linear equations with the same matrix (only the right-hand sides change); thus the relatively cumbersome problem of computing the inverse matrix is solved only once and the computations at all the subsequent approximations reduce simply to multiplying this matrix by a vector.

NUMERICAL METHODS IN SOLVING STEADY-STATE EQUATIONS OF GASDYNAMICS

O. M. Belotserkovskiy

In problems for solution on high-speed electronic computers one most frequently encounters systems of nonlinear differential equations, either ordinary or partial. While methods of numerical and asymptotic solution of systems of ordinary differential equations are well developed, intensive development of methods for solving partial differential equations started only with the advent of electronic computers. There exist three quite universal methods of solving systems of nonlinear partial differential equations, each of which utilizes a large number of directions and approaches and is suitable for the study of different classes of problems. /101

1. Method of finite differences. This is the most highly developed method of the three at the present time and is widely applied for solving linear and nonlinear hyperbolic, elliptical and parabolic type equations. Here, as a rule, the region of integration is subdivided into a fixed orthogonal grid. The derivatives of functions in the various directions are replaced by finite differences; usually a so-called implicit difference scheme is used. This results in the solution, at each step of the procedure, of a system of linear algebraic equations involving sometimes several hundred unknowns. Problems of the theory of difference schemes have been studied in sufficient detail for equations with constant coefficients.

2. Method of integral relations. In this method, which is a generalization of the well-known method of straight lines, the region of integration is subdivided into strips by a series of curves, the shape of which is determined by the form of the region's boundaries. The system of partial differential equations, written in divergence form, is integrated across these strips and then the functions contained in the integrands are replaced by certain interpolation expressions. The approximating system of ordinary differential equations thus obtained is integrated numerically. The method of integral relations, as the method of finite differences, is applicable to various types of equations. /102

3. Method of characteristics. This method is used only for solving hyperbolic type equations. The solution here is computed with the aid of a grid of characteristic lines, which is constructed in the course of computation. However, it is possible to use schemes of the method of characteristics, in which the computations are carried out by strips bounded by fixed lines.

The method of characteristics can be used for precise determination of the point of origin of secondary shock waves within the flow field as the result of intersection of characteristics of one family. However, if a large number of singularities occur, calculations by this method are difficult. This being so, this method should be used for computing hyperbolic problems in which the number of discontinuities is moderate (for example, steady-state problems of supersonic gasdynamics).

Lately, considerable success has been achieved in using the method of characteristics for the computation of three-dimensional gas flows.

Subsequently I shall not consider finite-difference methods (except for presenting several specific examples), since V. V. Rusanov, V. F. D'yachenko and G. I. Marchuk in their lectures have illuminated in sufficient detail many aspects of construction of difference schemes and of their applications. I shall dwell on certain questions of the use of the method of integral relations and lines, as well as of the method of characteristics for solving steady-state gasdynamics problems.

As is known, in gasdynamic problems one frequently has to deal with singularities of various kinds, discontinuities within the region of integration produced by the physical statement of the problem and nonlinearity of the differential equations. In this case the starting system of equations has no smooth solutions.

When the number of these singularities is small and it is possible to formulate certain boundary conditions for them, the problem reduces to determining continuous solutions in regions bounded by these singular lines or surfaces. In constructing numerical solutions of this class of problems, which appear, for example, in steady-state gasdynamics, extensive use is made, alongside difference methods, /1
of the method of integral relations and the method of characteristics, which will principally be discussed below.

In the case when the number of discontinuities inside the integration region is large, it becomes difficult to keep track of each of them. In order to cope with these difficulties, special difference schemes of "through" calculation were devised, which permit one to carry out calculations without regard for the singularities (this was discussed in detail in V. F. D'yachenko's lectures).

PART I. THE METHOD OF INTEGRAL RELATIONS

Since methods of numerical solution of ordinary differential equations are well developed, great practical importance is acquired by methods of solving partial differential equations which reduce them approximately to systems of ordinary differential equations. These methods include the method of straight lines and the method of integral relations. They, as the method of finite differences, are applied to equations of different types, including also mixed-type equations.

In the classical method of straight lines the integration region is subdivided into strips by fixed straight lines and the derivatives with respect to one of the directions are replaced by finite-difference relations (usually linear). This yields a system of ordinary differential equations, which is solved numerically. When using this approach, satisfactory accuracy is obtained by taking a quite large number of strips.

The method of integral relations, suggested by A. A. Dorodnitsyn [1] in 1951 for solving certain nonlinear problems of aerohydrodynamics, was developed by O. M. Belotserkovskiy and P. I. Chushkin [2] and has been used successfully for solving various problems of gasdynamics.

The idea underlying the method of integral relations consists in covering the region of integration by some curvilinear (in general) computational grid and in introducing into the starting system of equations a representation of functions with respect to one of the directions, which yields, in the two-dimensional case, an approximating system of ordinary differential equations, which is then solved numerically. The structure of the computational grids depends here on the shape of the boundaries of the region under study, and the starting system of partial differential equations is first written in the form of integral relations for the strips. For three-dimensional gasdynamic problems, one usually introduces an additional trigonometric approximation in the transverse plane. Functions can also be represented in two directions. /104

The freedom in selecting approximating functions makes it possible here to obtain a sufficiently exact solution with a small number of strips, which is of great importance in practical calculations. In the generalized method of integral relations [3], due to introduction of smoothing functions which are selected in accordance with the expected behavior of the sought functions, one gets a system of ordinary differential equations for functionals which are smoother than in the ordinary method of integral relations. This yields satisfactory accuracy with a relatively small number of approximating grid intersections.

As was pointed out, the method of integral relations has the advantage that here one uses the extensively developed tools of numerical solution of systems of ordinary differential equations. In addition, in the case of unbounded regions it is possible to use asymptotic methods for solving these equations. Programs for computer computations are found to be sufficiently simple and do not require a large machine memory. However, the application of this method involves difficulties when it is necessary to solve multiparameter boundary-value problems for the approximating system of higher order ordinary differential equations. In this case the method is effective if it yields a sufficiently exact solution when the system is of a rather low order or when the method of computations is satisfactorily organized.

For a number of years the Computation Center of the USSR Academy of Sciences and other organizations have engaged in the development and study of the method of integral equations, as well as its application to various problems of mechanics and physics. This method is surveyed in [2]. Here I shall briefly present the basic tenets of the method of integral equations and describe some features of its application. /105

1. The Fundamental Principles of the Method of Integral Equations

In this method the starting differential equations are written in divergence form. This form is suitable for representing differential equations of physics and mechanics expressing thelaws of conservation of mass, momentum, energy, charge, etc. In the two-dimensional case the starting differential equations in divergence form are taken in the following general form:

$$\frac{\partial}{\partial x} P_i(x, y, u_1, u_2, \ldots, u_k) + \frac{\partial}{\partial y} Q_i(x, y, u_1, \ldots, u_k) = F_i(x, y, u_1, \ldots, u_k), \quad i = 1, 2, \ldots, k, \tag{1.1}$$

where x and y are independent variables, $u_1, \ldots, u_k$ are the sought functions; P_i, Q_i and F_i are known functions of x, y, $u_1, \ldots, u_k$.

Let it be required to find the solution of system (1.1) in a region having the shape of a curvilinear rectangle with the boundaries

$$x = a; \quad x = b; \quad y = 0; \quad y = \Delta(x).$$

In individual cases it may happen that

$$a = -\infty; \quad b = -\infty; \quad \Delta(x) \equiv \text{const}.$$

We shall assume, concerning the boundary conditions of system (1.1), that there exists a total of k conditions at boundaries x = a and x = b, and k conditions at boundaries y = 0 and $y = \Delta(x)$. If boundary $y = \Delta(x)$ is not known a priori, then still another boundary condition is needed. If, however, the boundaries contain singular points, the corresponding boundary conditions may not exist — they are replaced by the requirement that the solution be regular. Multiplying each of equations (1.1) by some piecewise continuous function f(y) and integrating it with respect to y, from y = 0 to $y = \Delta(x)$, we get integral relations for i = 1, 2, ..., k (subscript i is omitted)

$$\frac{d}{dx}\int_0^{\Delta(x)} f(y) P dy - \Delta'(x) f(\Delta) P_\Delta + f(\Delta) Q_\Delta - f(0) Q_0 - \int_0^{\Delta(x)} f'(y) Q dy = \int_0^{\Delta(x)} f(y) F dy. \tag{1.2}$$

Here

$$P_\Delta = P[x, y, u_1(x, y), \ldots, u_k(x, y)]\,|_{y = \Delta(x)}.$$

Quantities Q_0 and Q_Δ have the same meaning.

We note that function f(y) may have a finite number of discontinuities of the first kind. Then $\int_0^{\Delta(x)} f'(y) Q dy$ should be replaced by $\int_0^{\Delta(x)} Q df$.

In the method of integral relations the solution is constructed in successive approximations. Let us consider the Nth approximation. The region of integration is subdivided into N strips, for which line N-1 is drawn between boundaries $y = 0$ and $y = \Delta(x)$ (Fig. 1.1):

$$y = y_n(x) = \frac{n}{N}\Delta(x), \quad n = 1, 2, \ldots, N-1,$$

where $y_0 = 0$, $y_N(x) = \Delta(x)$.

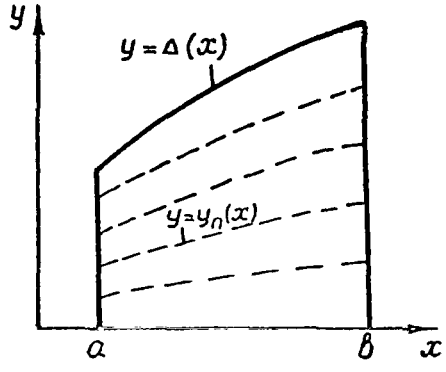

Figure 1.1.

Functions P, Q and F are now represented by certain interpolation formulas in terms of their values P_n, Q_n and F_n at the boundaries of strips $y = y_n(x)$. For example

$$P(x, y, u_1, \ldots, u_k) = \sum_{n=0}^{N} P_n(x) z_n(y), \tag{1.3}$$

where

$$P_n(x) = P(x, y_n, y_{1n}, \ldots, u_{kn});$$
$$u_{\nu n} = u_\nu(x, y_n), \quad \nu = 1, 2, \ldots, k, \quad n = 0, 1, 2, \ldots, N;$$

and $z_n(y)$ are interpolation functions, the specific form of which depends on the selection of interpolation formulas (i.e., on the form of initial functions). Functions Q and F are represented similarly.

We now select a system of groups of functions

$$\{f_n(y)\} \equiv \{[f_{11}(y)], [f_{21}(y), f_{22}(y)], \ldots \\ \ldots, [f_{N1}(y), f_{N2}(y), \ldots, f_{NN}(y)] \ldots\} \tag{1.4}$$

such that the Nth group contains N linearly independent functions. Functions of different groups may be identical, but as a whole the entire system of functions should be closed.

In the Nth approximation we now write integral relations such as (1.2) for each function $\{f_n(y)\}_N = \{f_{Nn}(y)\}$ (n = 1, 2, ..., N) of the Nth group of system (1.4). We will then have a total of kN linearly independent integral relations. Integrands P, Q and F are subsequently replaced by their corresponding interpolation expressions such as (1.3). Then the integrals contained in the integral relations will be written as

$$\int_0^{\Delta(x)} f_n(y) P dy \sim \Delta(x) \sum_{n=0}^{N} C_n P_n(x), \tag{1.5}$$

where C_n are numerical coefficients which depend on the selection of interpolation formulas and the form of functions $f_n(y)$ and are, in essence, generalized Coates coefficients.

Upon substituting Eqs. (1.5) into intergral relations such as (1.2), we obtain a system of ordinary differential equations in x (the so-called approximating system) for k(N + 1) unknown functions $u_{\nu n}(x)$ (ν = 1, 2, ..., k, n = 0, 1, 2, ..., N) on the boundaries of strips $y = y_n(x)$. This system is closed by k boundary conditions at the outer boundaries $y_0 = 0$ and $y_N = \Delta(x)$. The boundary conditions x = a and x = b complete the system of boundary conditions for the approximating system.

The approximating system of ordinary differential equations thus obtained is integrated by some numerical method on a computer.

We note that functions $P_n(x)$ enter Eq. (1.5) linearly. Hence the approximating system of ordinary differential equations will be linear in the derivatives dP_n/dx, and this means that it can be easily reduced to the computational form by simply solving it for these derivatives. This can also be done by computer.

In general, functions $f_n(y)$ are selected quite arbitrarily. We now consider the following two particular cases.

1. If the Dirac δ-function is assumed for $f_n(y)$

$$\{f_n(y)\}_N = \{\delta(y-y_n)\} \qquad (n = 1, 2, \ldots, N),$$

then we have the well-known method of straight lines, in which derivatives with respect to y are replaced by a finite difference expression satisfying the selected interpolation formulas (1.3).

2. If one assumes the step functions /108

$$\{f_n(y)\}_N = \left\{ \begin{array}{lll} 0 & \text{for} & y < y_{n-1} \\ 1 & \text{for} & y_{n-1} \leq y \leq y_n \\ 0 & \text{for} & y > y_n \end{array} \right\} \qquad (n = 1, 2, \ldots, N),$$

then the simple method of integral relations is obtained. Here the conservation laws expressed by Eqs. (1.1) will be written for the strips in the form of the integral relations

$$\frac{d}{dx} \int_{y_{n-1}}^{y_n} P\,dy - y_n' P_n + y_{n-1}' P_{n-1} + Q_n - Q_{n-1} = \int_{y_{n-1}}^{y_n} F\,dy.$$

A large number of gasdynamics problems has been solved using this scheme.

Yet another form of the method of integral relations is possible. In two-dimensional problems the integration region can be divided not into strips, but into subregions (Fig. 1.2), with the integration being carried out in two directions, thus approximating functions in two variables. The approximating system then becomes one of nonlinear algebraic or transcendental equations, which is solved numerically by computer. This scheme was used in a number of problems.

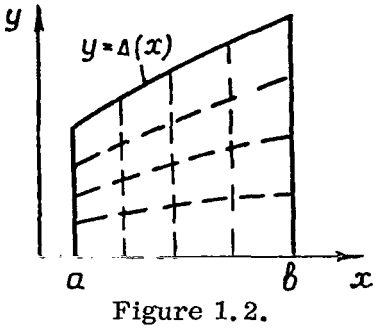

Figure 1.2.

Finally, the method of integral relations can be extended to the case of three-dimensional partial differential equations. Two approaches are possible here. In the first approach the initial system of equations is multiplied by some function which depends on two of the variables in question (the analog of function $f_n(y)$ in the two-dimensional case) and is integrated with respect to these two variables. Then, applying quadrature formulas to the double integrals thus obtained, the result is a system of ordinary differential equations with respect to the third variable. /109

In the second approach the initial system of equations is first integrated with respect to one of the variables and the integrands are represented in terms of this variable by interpolation expressions whose coefficients will depend on the two remaining variables. Then the approximating system becomes one of partial differential equations in these two variables and it can be solved by the method of integral relations developed for two-dimensional problems.

In the three dimensional case one may also combine the method of integral relations with that of finite differences or of characteristics.

2. Characteristic Features of the Method of Integral Relations and Some Problems of Its Practical Application

1. Let us emphasize the characteristic features of the method of integral relations and the advantages which are gained by them. The distinguishing feature of this method is the fact that here the solution of the problem is broken up into two separate stages. The first consists in reducing an exact system of partial differential equations to its approximating system of ordinary differential equations. Then, at the first stage, which can be considered separately, this approximating system is integrated numerically.

In this method it is actually an integral which is being approximated. This somewhat increases the approximation accuracy by decreasing the coefficient of the residual term. In addition, the integral is a smoother function than the integrand, due to which a satisfactory representation for the integral is obtained with a small number of interpolation nodes. Finally, the integral has continuous representation also in the case when the integrand has a finite discontinuity.

The starting equations in the method of integral relations are taken in divergent form, which is convenient and useful since integration with respect to one of the variables is then carried out exactly. In addition, as is easily seen, integral relations constructed from equations in divergence form and expressing conservation laws, remain valid also upon crossing the discontinuity surface. Thus, if the gasdynamic problem under study has several surfaces of discontinuity and it is difficult to keep track of all of them, the method of integral relations can be used for making through computations and then these surfaces can be constructed from gradients of "detached" quantities. /110

In the method of integral relations the integration region is in general subdivided by curved lines and the method proper is applicable to regions of arbitrary form — finite and infinite, with rectangular and curvilinear boundaries, as well as in the case when the boundaries are unknown. The latter is of particular significance in gasdynamics problems, where usually the shock wave shapes or boundaries of the region of influence are not a priori known.

Finally, in the method of integral relations one has available a quite arbitrary selection of interpolation expressions and functions $f_n(y)$ which must be selected with consideration of the nature of the solution. This makes it possible to obtain quite exact results with rather low-order approximations.

2. We now consider some aspects of practical application of the method of integral relations. Since this method is most extensively used in hydromechanics, our examination will concern itself with these applications.

When using this method the starting system of partial differential equations can, in general, be written in any coordinates. However, it is more convenient to use an orthogonal coordinate system in which one of the curvilinear boundaries, i.e., the body's contour, is a coordinate curve. Thus, elliptical coordinates should be used for elliptical contours. In considering contours of general form it is convenient to use the s, n coordinate system, where s is the arc length measured along the contour and n is the normal to the contour. In some schemes of solution using this method it becomes necessary to make some two boundaries parallel, let us say, boundaries $r = r_0(\theta)$ and $r = r_1(\theta)$. This can be done by introducing the coordinates /111

$$\xi = \frac{r - r_0}{r_1 - r_0}.$$

If the integration region is infinite in the physical plane, the former can be made finite by converting to some new independent variable.

As far as possible, the starting system written in divergence form should contain known integrals instead of some differential equations. This makes it possible to reduce the number of differential equations and thus to reduce the number of computations. It should be noted that mathematically equivalent systems are not always equivalent from the point of view of numerical solution of their corresponding approximating systems. Hence the starting system should retain those equations which typify the nature of the phenomenon. For example, in calculating the supersonic flow around a cone at an angle of attack, when one uses the spherical coordinates r, θ and ψ, the Bernoulli integral should be introduced instead of the momentum equation in the projection on r, i.e., in the direction of the main flow, since the two other equations of motion describe a finer and more important (in this problem) effect, which is the transverse flow.

Selection of the direction of approximation and its form is of great importance in constructing the approximating scheme in the method of integral relations. It is found that the solution obtained here yields a more exact distribution of the sought functions in the direction of integration of the approximating scheme than in the direction along which it is approximated. This should be taken into account in selecting the approximation direction, since the functions may vary in one direction more sharply than in the other. In all cases the region of influence of the approximating system should coincide with or be larger than the region of influence of the starting system.

The integration region can be subdivided into strips either by equidistant lines, or in the zone of sharp variation of functions the intermediate lines can be made denser. Usually use is made of "through" interpolation using values of generating functions at the boundaries of all the strips (Fig. 1.3a). It is also possible to use "piecewise" interpolation. For example, the functions can be represented linearly across each individual strip (Fig. 1.3b), and parabolically across adjoining strips /112

(Fig. 1.3c). This permits considerable simplification of the form of the approximating systems, with the necessary accuracy provided by using a large number of strips. Such approximations are convenient in solving Cauchy problems (computation of boundary layers or flows in the supersonic region), and they may be found more reliable than the "through" approximations in regions with weak discontinuities. It is also possible to use piecewise overlapping approximations (Fig. 1.3d).

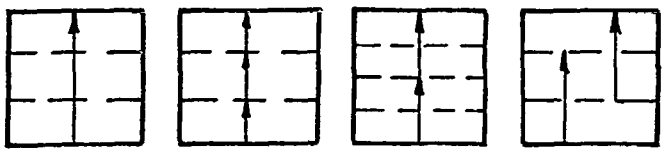

Figure 1.3.

It should be mentioned that the system of base functions of the selected interpolation formulas should be a Chebyshev system. Usually polynomial or trigonometric interpolation is used. The interpolation formulas should make allowance for the nature of variation of the functions they represent, in particular their symmetry, behavior at singular points and their asymptotic behavior. For example, in the problem of subsonic flow around ellipses, which was solved in the elliptical coordinates ξ, η, it is possible to use both polynomial and trigonometric approximation with respect to η. But only in the latter case is it possible to exactly satisfy the conditions at infinity.

In describing the generalized method of integral relations above we mentioned the requirements imposed on the system of functions $\{f_n(y)\}$. It should also be added that if the integrands have a singularity in the integration region, then functions $f_n(y)$ should naturally be selected so as to ensure convergence of all the integrals in the integral relations. Functions $f_n(y)$ play the role of "weights" for integral relations. We also note that the use of functions $f_n(y)$ facilitates the reduction of the approximating system to canonical form.

The approximating system of ordinary differential equations is, as a rule nonlinear and is integrated numerically by computer using standard programs. No difficulties are encountered here in the case of the Cauchy problem, while the solution of boundary-value problems is either reduced to iterative solution of the Cauchy problem and selection of boundary conditions by interpolation (which is done by the machine) or by applying iteration schemes (Newton's generalized method, etc.). /113

In some cases one must construct the solution in the vicinity of singularities. Regular singularities are crossed, for example, using appropriate series. If, however, the region is not bounded and a part of the boundary conditions is specified at infinity, then one determines the asymptotic behavior of the approximating

system, which is then "spliced" to the numerical solution in the finite part of the region.

Analytic estimates of the convergence and accuracy of the method of integral relations can be obtained for second order linear equations, their systems, as well as for some nonlinear hyperbolic [4] and mixed type nonlinear equations.

Let us, for example, consider the problem which models a flow with a detached bow shock [5],

$$\frac{\partial^2[(1-x)u]}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

or its equivalent system

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0;$$

$$\frac{\partial[(1-x)u]}{\partial x}+\frac{\partial v}{\partial y}=0$$

with the boundary conditions

$$(0\leq x\leq 1,\ 0\leq y\leq 1):$$

$$u(0,y)=0,\quad v(x,0)=0,\quad u(x,1)=x.$$

If the region of integration is subdivided by lines y = const and then linear interpolation is applied, the corresponding approximating operator will be written as an inhomogeneous system of linear differential equations with constant coefficients. The solution of this system is found analytically in any approximation in the form of finite series.

The maximum magnitude of error (the difference between the exact and approximate solution) will be of order o $(N^{-4/3})$, whence convergence of the method in the given case follows as $N\to\infty$ (N is the number of the approximation). /114

The main accuracy criterion of the method of integral relations in actual computations is numerical estimates consisting of: 1) making calculations in different approximations; 2) making calculations using different numerical schemes and utilizing approximate representation of functions in diametrically opposite directions; 3) checking of exact integrals, properties of individual exact solutions, etc., which are not used in constructing the starting system.

Experience in solving various gasdynamics problems by the method of integral relations shows that in the majority of cases a rationally constructed scheme yields sufficient accuracy as early as the second or third approximations.

3. Solution of Gasdynamics Problems by the Method of Integral Relations

The method of integral relations has been employed for solving a large number of gasdynamic problems. This pertains primarily to calculation of two-dimensional (flat, axisymmetrical, conic) steady gas flows. The problems considered pertained to internal and external flows in potential motion, vortex flow of ideal or real gases past bodies in the presence of shock waves, calculation of the boundary layer at bodies in a flow of viscous gas. Equations describing these gas flows are of different types, i.e., they may be parabolic, elliptical, hyperbolic and mixed. In addition the method of integral relations is presently used with great success in computation of three-dimensional mixed and supersonic flows of gases about bodies of revolution at an angle of attack, and also in the case of unsteady flows.

We shall first consider as an illustration the flow of gas at the speed of sound past a body. The generalized method of integral relations will be illustrated by means of boundary layer problems and, finally, specific numerical schemes (including computational results) will be constructed for calculating the flow past blunt bodies with a detached bow shock wave. /115

1. Symmetrical flow about bodies moving at the speed of sound

The method of integral relations for solving this mixed-type problem was developed by P. I. Chushkin [6], who considered the sonic gas flow (incident Mach number $M_\infty = 1$) about flat and axisymmetrical bodies with an elliptical as well as arbitrary contour. The solution is constructed in the ξ, η, orthogonal coordinates, in which

$$\xi = r - r_0(\theta); \quad \int \frac{d\eta}{r_0'(\eta)} = -\frac{1}{r} + \int \frac{d\theta}{r_0'(\theta)},$$

where $r = r_0(\theta)$ is the equation of the body's contour in polar coordinates. Accordingly, curve $\xi = 0$ represents the body's contour, while at infinity these coordinates tend toward the polar coordinate system. For the case of an elliptical contour these coordinates are simply replaced by ordinary elliptical coordinates.

One can isolate in the flow under study (of which only the upper half-plane is considered) a minimal region of influence, whose boundaries are the axis $\eta = 0$, a part of the body's contour and the semi-infinite characteristics $\eta = \eta_1(\xi)$, whose shape is not a priori known and which is tangent at infinity to the line $M = 1$ (Fig. 1.4). The flow in this region is potential and is described, for example, in the two-dimensional case, by the system of equations

$$\frac{\partial \chi}{\partial \xi} + \frac{\partial \omega}{\partial \eta} = 0; \quad \frac{\partial \lambda}{\partial \xi} - \frac{\partial \mu}{\partial \eta} = 0, \qquad u = w_\xi, \quad v = w_\eta; \qquad /116$$

$$\chi = H_{II}\,\rho u; \quad \omega = H_I\,\rho v; \quad \lambda = H_{II}\,v; \quad \mu = H_I\,u$$

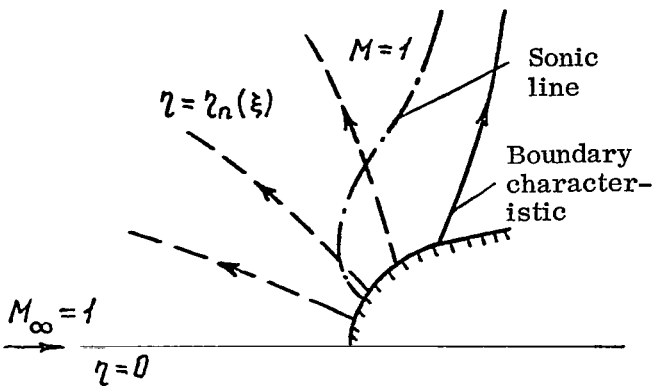

Figure 1.4.

and the Bernoulli integral; here H_I and H_{II} should be treated as Lamé parameters for orthogonal coordinates assumed.

The nature of these equations for the sonic line is different. The boundary conditions consist of the symmetry requirement at axis $\eta = 0$, the requirements that the normal velocity component at the body become zero and that the sonic flow be uniform at infinity; in addition, the differential compatibility condition should be satisfied at the bounding characteristic.

In solving this problem the region of influence is subdivided by the lines

$$\eta_n(\xi) = \frac{N-n+1}{N} \cdot \eta_1(\xi), \quad n = 2, 3, \ldots, N,$$

which pass between the bounding characteristic and the axis (see Fig. 1.4). Integrands χ and λ in the integral relations are represented with consideration of symmetry by the following interpolation polynomials:

$$\chi = \sum_{n=0}^{N} a_n \cos\left(n \frac{\pi}{2} \cdot \frac{\eta}{\eta_1}\right), \quad \lambda = \sum_{n=1}^{N} b_n \sin\left(n \frac{\pi}{2} \cdot \frac{\eta}{\eta_1}\right).$$

The system of differential equations for ξ will have only one singularity at infinity which must be entered if integration is carried out starting from the body. Approximations of this kind ensure the existence, at infinity, of a singularity corresponding to uniform sonic flow, and thus make it possible to satisfy the impermeability condition along the entire contour of the body. Thus, the solution in any approximation is determined upon exact satisfaction of all of the problem's boundary conditions specified in exact consideration of the region of influence.

2. Flow in the boundary layer

The generalized method of integral relations was used by A. A. Dorodnitsyn and Yu. N. Pavlovskiy [2, 3, 7] for calculation of incompressible as well as compressible laminar boundary layers at plane and axisymmetrical bodies with allowance for heat conduction and radiation.

The plane problem for an incompressible fluid in the method of integral relations reduces to solution of a single equation of the form

$$\frac{d}{d\xi}\int_0^1 \theta u f(u)\,du = \frac{\dot{U}}{U}\int_0^1 \theta(1-u^2)f'(u)\,du - \frac{f'(0)}{\theta_0} - \int_0^1 \frac{1}{\theta} f''(u)\,du,$$

where ξ, η are Dorodnitsyn's variables; $\dot{U} = \frac{dU}{d\xi}$; U is the velocity at the outer periphery of the boundary layer; u is the tangential velocity component, referred to $U(0 \le u \le 1)$; θ is a quantity inversely proportional to the drag coefficient (the sought function), upon determining which it is possible to find all the boundary layer characteristics of interest; f(u) is a "weighting" function. The requirement put to f(u) is that it tend sufficiently rapidly to zero as $u \to 1$ and that then the integrals be converging. /11

It is known that as $u \to 1$, $\theta = O\left(\frac{1}{1-u}\right)$, consequently $\{f_n(u)\}$ can be represented by a system of power-law functions (in the N-th approximation):

$$\{f_n(u)\}_N = \{(1-u)^n\} \qquad (n = 1, 2, \ldots, N).$$

The integration region $0 \le u \le 1$ is subdivided into strips by lines u_n = const, and the integrands are approximated as

$$\theta = \frac{1}{1-u}\sum_{n=0}^{N-1} A_n u^n; \quad \frac{1}{\theta} = (1-u)\sum_{n=0}^{N-1} B_n u^n.$$

The integral relations constructed for each of the functions $f_n(u)$, upon substitution in them of the above representations, yield a system of ordinary differential equations in ξ, for which the Cauchy problem is solved, for determining θ_n (n = 0, 1, 2, ..., N - 1).

Algorithms of numerical schemes of the method of integral relations will be described in more detail later on, in examining problems of supersonic flow past blunt bodies and in calculating boundary-layer flows.

PART II. THE METHOD OF CHARACTERISTICS

The method of characteristics is used extensively for calculation of hyperbolic problems of mechanics of continuous bodies. These are unsteady and steady problems of gasdynamics, the theory of elasticity, etc.

We shall not consider in this review numerical schemes developed for calculating unsteady one-dimensional problems, since this was discussed in the preceding lectures. Here we shall be concerned with construction of algorithms of the method of characteristics for computation of two- and three-dimensional supersonic steady gas flows. These include flows inside of the nozzles, as well as various problems of external flow. We shall consider flows of an ideal gas with constant specific heats, as well as flows with chemical reactions and radiation. /118

1. Development of the Method of Characteristics

The method of characteristics for numerical integration of hyperbolic equations was suggested as far back as over sixty years ago by Massau. Specialists in gasdynamics developed the method of characteristics virtually anew and primarily for calculation of steady-state two-dimensional and unsteady one-dimensional gas flows. Important contributions were made by F. I. Frankl, R. Courant, K. O. Friedrichs, A. Ferri, A. A. Dorodnitsyn and others.

The advent of computers made it possible to sharply increase the number of computational points and at the same time to use the method of characteristics in its most general form for computation of supersonic gas flows with physical and chemical transformations, and most recently for calculation of general three-dimensional flows. A large contribution to the development of the modern method of characteristics was made by Yu. D. Shmyglevskiy, O. N. Katskova, F. E. Ehlers, P. I. Chushkin, V. V. Rusanov, A. N. Krayko, I. N. Naumova, K. M. Magomedov, V. B. Minostsev and others [8-20].

A few words on the properties of characteristics. As is known, weak disturbances in hyperbolic regions propagate in the flow along certain lines (surfaces), which are termed characteristic. These are Mach lines in two-dimensional flow and conical-type surfaces (conoids) in three-dimensional flows. A line (surface) is a characteristic if it is impossible to uniquely define the derivatives of all the sought functions on this line (surface). This is due to the impossibility of solving the Cauchy problem with boundary conditions for such a surface. It can be shown that using some linear combination of starting equations for the hyperbolic region one can attain a situation whereby these equations will contain only internal derivatives along characteristic surfaces, and will not contain derivatives which lead us beyond the ensemble of characteristics.

The use of this mathematically equivalent system for constructing the numerical algorithm has its advantages over other numerical methods: the equations to be solved become much simpler at the characteristic surface; the solution's domain of dependence is exactly taken into account in using the characteristic grid; in addition, the method of characteristics has a high degree of mathematical rigor (the existence of a solution and convergence were proved for it). Due to these /119

circumstances the method of characteristics has been extensively used in solving hyperbolic problems.

Numerical schemes of the method of characteristics in two- and three-dimensional problems are constructed as purely characteristic schemes (the region of integration is covered by a curvilinear characteristic grid), as well as numerical schemes in which computations can be carried out "by strips." We shall attempt to describe these approaches below. We start with presentation of the two-dimensional method of characteristics for solving steady supersonic problems of gasdynamics in the general case of nonequilibrium flows.

2. Two-Dimensional Characteristics

1. Nonequilibrium flows

Following A. N. Krayko [12] we consider steady-state plane ($\nu = 0$) or axisymmetric ($\nu = 1$) supersonic flow of an inviscid and thermally nonconducting gas in the nonequilibrium case. This flow is described by continuity, motion and energy equations

$$\rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \frac{\nu \rho v}{y} + \frac{d\rho}{dt} = 0; \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0;$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0; \quad \frac{dh}{dt} = \frac{1}{\rho} \frac{dp}{dt},$$

which interrelate the velocity components u and v (along the x and y axes respectively), density ρ, pressure p and specific enthalpy h; x, y denote a Cartesian coordinate system.

Let the thermodynamic state of the gas be defined by the pressure p, temperature T of the translational degrees of freedom of some gas component, and n variables q ($q_1, \ldots, q_n$) describing irreversible processes. The mass concentrations, energies of internal degrees of freedom, etc. can be used as variables q. We postulate that the variation of q is represented by the expression /120

$$\frac{dq_i}{dt} = F_i(p, T, q) = \varphi^i(p, T, q) f_i(p, T, q) \qquad (i = 1, \ldots, n),$$

where F_i, φ^i and f_i are known functions of p, T and q; functions φ^i are related to the rates of the physio-chemical processes; here the applicable variables q_i are "frozen" for $\varphi^i \equiv 0$ and are at equilibrium for $\varphi^i \equiv \infty$. In accordance with this the equilibrium values of q_i are defined from $f_i(p, T, q) = 0$. These equations describe a wide range of processes: chemical reactions, excitation of internal degrees of freedom, change in translational temperatures of various gas components, etc.

Let the equation of state of the gas $\rho = \rho(p, T, q)$ and an expression for the specific enthalpy $h = h(p, T, q)$ be also known.

The starting equations will be solved in the Cartesian coordinates x, y. We shall use dimensionless variables, taking as the characteristic reference variables some linear dimension, velocity V_∞ and density ρ_∞ (in internal problems these can be the velocity and density of the incident flow), as well as the gas constant of some gas $\Re_\infty$. Variables q should be reduced to the dimensionless form with consideration of their physical meaning.

Let us take as the principal unknown functions the following quantities: $\zeta = \operatorname{tg}\vartheta$; $\beta = \sqrt{\frac{V^2}{a^2} - 1}$, where ϑ is the angle made by the velocity vector with the x axis,* c is the "frozen" speed of sound, defined as

$$c^{-2} = \rho_p + \frac{\rho_T}{h_T}\left(\frac{1}{\rho} - h_p\right).$$

Here and in the following, letter subscripts of ρ and h will dentoe the applicable partial derivatives.

In the case of supersonic flow there exist three families of real characteristics: /121
two families of Mach lines (characteristics of the first and second families) and the streamlines ($\psi = \text{const}$).

The characteristics of the first (using the upper sign in the equations) and second (lower sign) families are defined by the expressions

$$dy = \frac{\beta\zeta \pm 1}{\beta \mp \zeta} dx = B^{\pm} dx; \tag{2.1}$$

$$d\psi = \pm \frac{\rho V y^{\nu} (1+\zeta^2)^{1/2}}{\beta\zeta \pm 1} dy; \tag{2.2}$$

$$\frac{1}{1+\zeta^2} d\zeta \pm \frac{\beta}{\rho V^2} dp \pm \frac{1}{\beta\zeta \pm 1}\left[\nu \frac{\zeta}{y} - \frac{(1+\zeta^2)^{1/2}(\rho_T h_{q_i} - h_T \rho_{q_i}) F_i}{h_T \rho V}\right] dy = 0, \tag{2.3}$$

with tensor notation used in the last equation and summation over i understood (for repeated subscripts): $\nu = 0$ or 1, respectively, for the two-dimensional or axisymmetric cases.

*It is assumed that in the flow region under computation $\vartheta \neq \pi/2$ (if the contrary is true, then ζ is replaced by the expression $\zeta_1 = \cot\vartheta$).

We note that special cases may arise in using these equations when the angles of inclination of characteristics of the first and second families will either tend to $\pi/2$ (i.e., $\beta-\zeta\to 0$ and $\beta+\zeta\to 0$), or tend to zero (i.e., $\beta\zeta+1\to 0$ and $\beta\zeta-1\to 0$). In these cases the possible singularities can be eliminated by multiplying the equations by appropriate multipliers.

The following relations are satisfied at the streamlines

$$dq_i = \frac{F_i(1+\zeta^2)^{1/2}}{V}dx \equiv C_i\,dx; \tag{2.4}$$

$$dT + Ddp + E_i\,dq_i = 0; \tag{2.5}$$

$$\frac{V^2}{2} + h = h_0(\psi),$$

where $D = \frac{\rho h_p - 1}{\rho h_T}$; $E_i = \frac{h'_{q_i}}{h_T}$; $h_0(\psi)$ is the total enthalpy of the gas which is a known function of ψ. The sought functions here are x, y, β, ζ, ψ, p, T and q.

For a perfect gas $\left(\varkappa = \frac{c_p}{c_v} = \text{const}\right)$ these equations have the form [8-9]: /122

$$dy = \frac{\beta\zeta \pm 1}{\beta \mp \zeta}dx; \tag{2.6}$$

$$d\zeta \mp \frac{2\beta^2(1+\zeta^2)}{(\varkappa+1)(1+\beta^2)(1+k\beta)}d\beta \pm \nu\frac{\zeta(1+\zeta^2)}{\beta\zeta\pm 1}\frac{dy}{y} \mp \frac{\beta(1+\zeta^2)}{\varkappa(\varkappa-1)(1+\beta^2)}dS = 0; \tag{2.7}$$

$$d\bar{\Psi} = \pm y^{\nu}\frac{\sqrt{(1+\beta^2)(1+\zeta^2)}}{(\beta\zeta\pm 1)\left(\frac{1}{k}+\beta^2\right)^{1/(2k)}}dy; \tag{2.8}$$

$$S = S(\psi),$$

where α is the Mach cone angle, $k = \frac{\varkappa-1}{\varkappa+1}$, $\bar{\Psi}$ is a function related to the streamline function ψ as

$$d\bar{\Psi} = \varkappa^{1/(\varkappa-1)} \cdot e^{S/(\varkappa-1)}\,d\psi; \tag{2.9}$$

and $S = \ln\frac{p}{\rho^{\varkappa}}$ is the entropy. The sought functions here are x, y, β, ζ, $\bar{\Psi}$, and S. Although these equations appear to be more complicated than those usually used, where θ = arc tan W_x/W_y and α are used instead of β and ζ, they do not contain trigonometric functions, which provides a substantial economy (several fold) of machine time.

In using the method of characteristics one has to solve a number of elementary problems: the point in question here may have to be computed in the field on the body, on the shock wave, etc. Here the sought point lies at the intersection of lines which are definied either by differential or finite [difference] relations. These lines may for example, be characteristics, streamlines, shock waves, etc. The advantage of the method of characteristics consists in the fact that the characteristic equations contain only derivatives along the characteristics proper; these are written in difference form along the characteristic grid and the entire field of flow is computed step by step.

Usually one uses a method of characteristics in which a differential equation in the form

$$\Sigma A_i dZ_i = 0,$$

which is valid at some line ab is replaced by a finite difference equation of second-order accuracy

$$\frac{1}{2}\sum_i (A_{ia} + A_{ib})(Z_{ib} - Z_{ia}) = 0.$$

The system of algebraic equations thus obtained is not solved explicitly, but is transformed for numerical computation to a form convenient for calculation by the method of successive approximations. /123

Two schemes of the method of characteristics have been most completely developed. In the first [9] the characteristic grid is constructed in the course of computations as the intersection of characteristics of two different families, i.e., the (x, y) coordinates of the points of intersection are calculated simultaneously with the flow variables. In the second approach [14] the calculation is carried out "by strips", bounded by lines x_0 and $x_0 + \Delta x$; here the segments of characteristics within each strip are drawn in such a manner that at $x_0 + \Delta x$ they pass through points with some given value of y.

2. Solution of elementary problems

We shall now consider briefly the solution of elementary problems using both schemes having, for simplicity, reference to a perfect gas ($\varkappa = c_p/c_v = \text{const}$).

Scheme I (characteristic grid) [8-13]

1. Calculation of a point in a field. To determine quantities $\{x, y, \beta, \zeta, \psi\}$ in point 3 from known values in points 2 and 1, Eqs. (2.6)-(2.8) are written in difference form along characteristics 2-3 and 1-3 (Fig. 2.1). The magnitude of S, the entropy, in point 3 is found from $S=S(\psi)$, which relationship is determined at the shock wave, while the value of function ψ is determined in terms of $\bar{\psi}$ from Eq. (2.9). The system of difference equations [thus obtained] is solved by iterations; here in the first iteration the values in point 3 are determined from those in point 2 (or 1), and then average values between points 2-3 or 1-3 are taken. Usually three iterations are sufficient.

The situation is more complicated in the nonequilibrium case [12]. Quantities $\{x, y, \psi, p, \zeta\}$ in point 3 are found from Eqs. (2.1)-(2.3) written in difference form. Knowing ψ_3, we find point 4 as the point of intersection of the streamline (arriving at point 3) with a known segment of characteristic 1-2' (Fig. 2.2). The variables in point 4 are determined by quadratic interpolation for the value $\psi_4 = \psi_3$ from known data on characteristic 1-2'. Then, from Eqs. (2.4)-(2.5) written in difference form along streamline 3-4,

$$q_{i3} = q_{i4} + C_i(x_3 - x_4),$$

$$T_3 = T_4 - D(p_3 - p_4) - E_i(q_{i3} - q_{i4})$$

/124

we find $\{q_i, T\}$ in point 3. And then from equations

$$\rho = \rho(p, T, q); \quad h = h(p, T, q);$$

$$c^{-2} = \rho_p + \frac{\rho_T}{h_T}\left(\frac{1}{\rho} - h_p\right); \quad \frac{W^2}{2} + h = h_0(\psi)$$

we find $\{\rho, h, c, W\}$ in point 3.

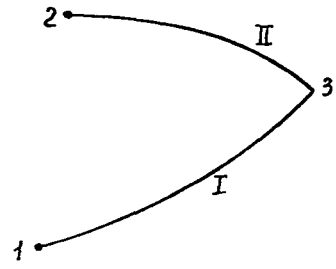

Figure 2.1.

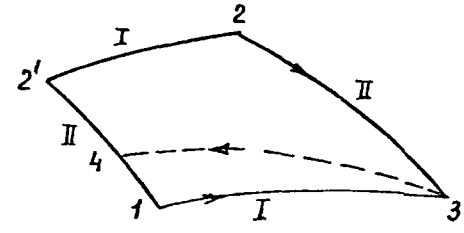

Figure 2.2.

As equilibrium is approached the convergence of this method detoriates (due to the difference representation of Eq. (2.4)), requiring a significant reduction in the integration step, and this involves large expenditure of machine time. The point is that the equation for C_i contains the quantity $F_i = \varphi^i f_i$ and close to equilibrium the values of f_i are close to zero and are very sensitive to changes in temperature T and in corresponding variables q_i (a small error in determining T_3 and q_{i3} at the preceding iteration results in a large error in the following iteration

for q_{i3}, which leads to increased error in determining T_3, etc.,). To eliminate this, f_{i_3} is expanded in a power series in $q_{i3} - q_{i4}$ (the two first terms of the series suffice). This difference form retains second-order accuracy, converges independently of the state of the flow and tends to the corresponding equilibrium state as $\varphi^i \to \infty$. The computation of purely equilibrium states is also not free of difficulties. This is discussed in more detail in [10, 11, 13].

2. A point on a body with contour $y = f(x)$ is also computed by iterations, using the impermeability condition, here in each iteration the body's contour is approxi- /125
mated by the tangent at point 1^i, where $x_1^i = x_3^i$. The unknown functions here are x, y and β (Fig. 2.3).

3. Point 3 at shock wave 0-3 is computed by selecting the tangent of the angle made by the shock wave (tan σ), from satisfaction of the compability condition at characteristic I of family 1-3, using known relationships across the shock (Fig. 2.4).

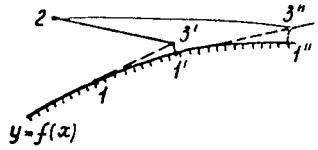

Figure 2.3.

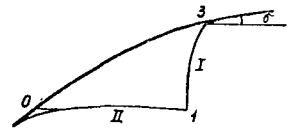

Figure 2.4.

Scheme II (computation "by strips") [13, 14]

This scheme is illustrated by axisymmetrical flows of a perfect gas. Let the body's contour be specified as $y = y_T(x)$. The solution will be obtained in variables x, ξ, where $\xi = \dfrac{y - y_T}{y_B - y_T}$ ($y = y_B(x)$ is the equation of the shock wave).

Using again the sought functions ζ, β, S and eliminating y from the preceding equation, we get

$$y = (y_B - y_T)\xi + y_T$$

and instead of Eq. (2.8) we shall use

$$\frac{d\xi}{dx} = \frac{1}{y_B - y_T}[\zeta - (y_B' - y_T')\xi + y_T'].$$

The computing scheme is described below.

Let the values of sought functions at a number of points $\xi = \xi_i = \text{const}$ (i =..., n - 1, n n + 1, ...) be known on some strips $x = x_0$ located between the shock wave ($\xi = 1$) and the body ($\xi = 0$) in a supersonic flow. It is required to determine the values of the sought functions for the same values of $\xi = \xi_i$, but in the strip $x_0 + \Delta x$. The same elementary problems are [then] solved, but in variables (x, ξ).

The computation is started from a specific point 3 on the shock wave and is carried out by selecting values of $\tan \sigma = y'_B$ (contained in the equation) so as to satisfy the compatibility condition at characteristic I of family 1-3 (Fig. 2.5a).

In calculating point 3 inside the field the values of the sought functions are specified in it approximately, and then characteristic I of family 1-3, characteristic II of family 2-3 and streamlines 4-3 are extended outward from this point. The values of sought functions in points 1, 2 and 4 are determined by interpolation, and then the quantities in point 3 are adjusted (Fig. 2.5b). /126

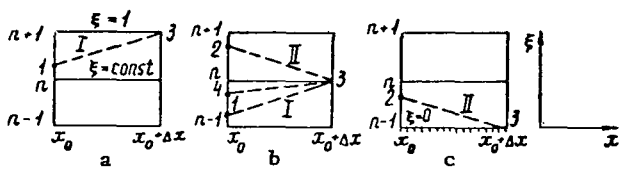

Figure 2.5.

To calculate the body's contour in point 3 (single unknown quantity β_3) one uses the appropriate difference equations along characteristic II of family 2-3 (Fig. 2.5c).

In all cases the values of sought functions in the intermediate points 1, 2 and 4 at strips x_0 are found by quadratic interpolation between the points n - 1, n, n + 1. The magnitude of step Δx is coordinated with that of $\Delta\xi$ on the basis of stability and accuracy of the computational scheme.

This scheme requires somewhat more machine time than scheme I, but has a number of advantages: a) here the coordinates of points of intersection are known from the start and need not be stored in the computer memory; b) the computational grid can be easily changed in the process of computations, which makes it possible to select in the most advantageous manner; c) the starting data are here read out by sections x = const, which is very convenient in practice; d) this scheme can be conveniently combined with the finite-difference method, for example in computing three-dimensional problems.

When using the method of characteristics the computations become complicated near the sonic line. Here it is necessary to first move away from this line by some

other methods (for example, by series), and only then apply the method of characteristics. However, these problems are not discussed here, since they are quite satisfactorily examined in previously mentioned references. Below we present examples of calculations using the method of characteristics. More detail can be found in the cited references.

3. Some calculated results /127

The method of characteristics has been widely used to solve a large variety of steady-state two-dimensional supersonic gasdynamics problems. These pertained to flows inside nozzles, as well as flows past bodies. Ideal gas flows, as well as equilibrium and nonequilibrium flows of real gases were considered. These problems are best solved by computer using the method of characteristic when using specially compiled sufficiently general programs. The latter should be suitable for computing a given class of problems (nozzle flow, flow past bodies) for different geometric parameters, incident-flow Mach numbers and for gases with different thermodynamic properties. To illustrate the method of characteristics we now present some results for supersonic gas flows obtained in the Computing Center of the USSR Academy of Sciences.

O. N. Katskova and Yu. D. Shmyglevskiy [9] computed the axisymmetrical flow of a freely-expanding perfect gas with a plane transition surface. This flow is partially attained in a nozzle if its generatrix has a sharp corner in the throat cross section, and occurs in a region covered by a fan-shaped pattern of characteristics, emanating from this sharp corner. The solution in the immediate vicinity of the transition surface was found analytically by series expansion, while the method of characteristics was applied to the remaining part of the nozzle. Detailed tables were computed for this axisymmetric flow for an ideal gas with different values of the ratio of specific heats.

O. N. Katskova [11] also extended the solution obtained in [9] to the case of a real gas in thermal equilibrium. Figures 2.6-2.7 show some computed results for this case. The real gas considered was hydrogen, the temperature and pressure at the transition surface were taken as 5000°K and 1 atm. Figure 2.6 depicts three different streamlines, while Fig. 2.7 shows the variation in M along the flow axis. Here the solid lines denote results for equilibrium flow of hydrogen, while the dashed lines show, for comparison, results of calculations for a perfect gas with various values of the ratio of specific heats $\varkappa$.

Still another example of computations using the method of characteristics in- /128
volves nonequilibrium flow of dissociating oxygen in the expanding part of axisymmetric nozzles with a sharp corner in the contour. These calculations were carried out by O. N. Katskova and A. N. Krayko [12]. A diagram of this type of nozzle is shown in Fig. 2.8. Here AC is the nozzle contour, AD is the initial and AB is the final characteristic of the acceleration portion of the nozzle (i.e., of the fan of rarefaction waves emanating from point A in the aforementioned corner), and BC is the closing characteristic. It was assumed that the flow in the initial section (x = 0) is at equilibrium, with the following constant variables: /129
$T_* = 5000°K$, $p_* = 1$ atm, $M_* = 1.001$. The constant direction $\vartheta = 0°$ of the velocity vector was specified at characteristic BC, and M = 5 was given in point B.

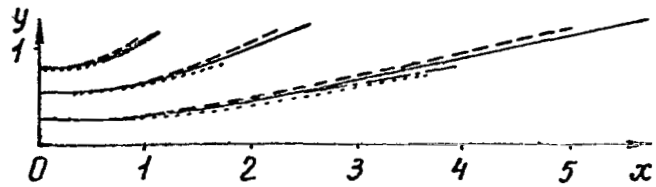

Figure 2.6.

• • • $\varkappa = 1{,}667$; – • – $\varkappa = 1{,}4$; – – – $\varkappa = 1.14$; ——— real gas.

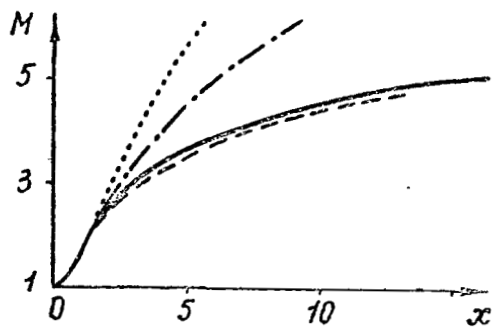

Figure 2.7.

• • • $\varkappa = 1{,}667$; – • – $\varkappa = 1{,}4$; – – – $\varkappa = 1{,}14$; ——— real gas.

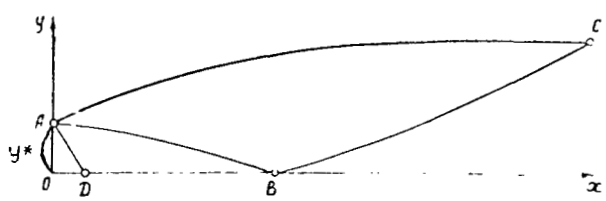

Figure 2.8.

The calculated results are shown in Figs. 2.9-2.12. Here the continuous lines pertain to nonequilibrium flow (here numbers 1, 2, 3 denote respectively the results for nozzles with initial-section radii $y_* = 0.5$, 2 and 8 cm), the dashed lines pertain to frozen flow (degree of dissociation q is constant), and the dash-dot lines are for equilibrium flow. All these calculations were made for the same temperature, pressure and inlet velocity. Changes in the relative temperature T/T_* and in q along the axis of accelerating segment are shown in Figs. 2.9 and 2.10, respectively, while the variation of the same quantities along the nozzle walls is depicted in Figs. 2.11 and 2.12. The graphs show substantial differences in results for the nonequilibrium, frozen and equilibrium flows.

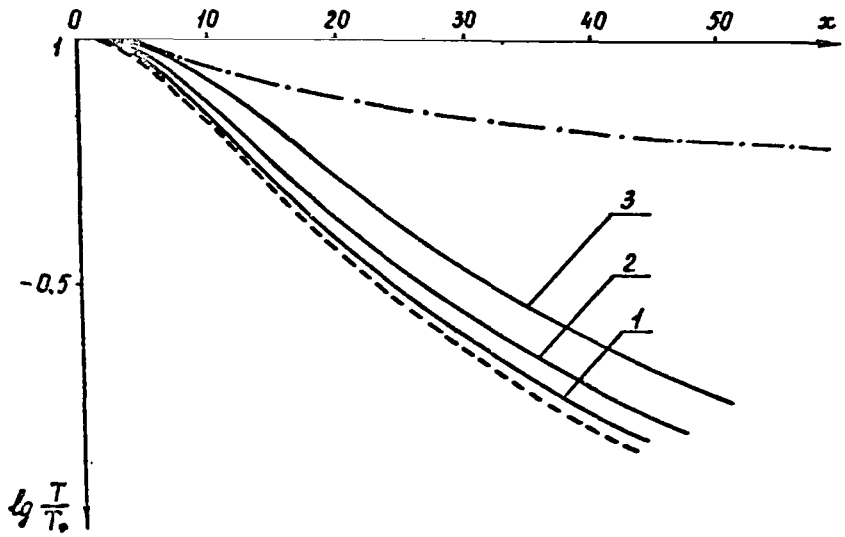

Figure 2.9.

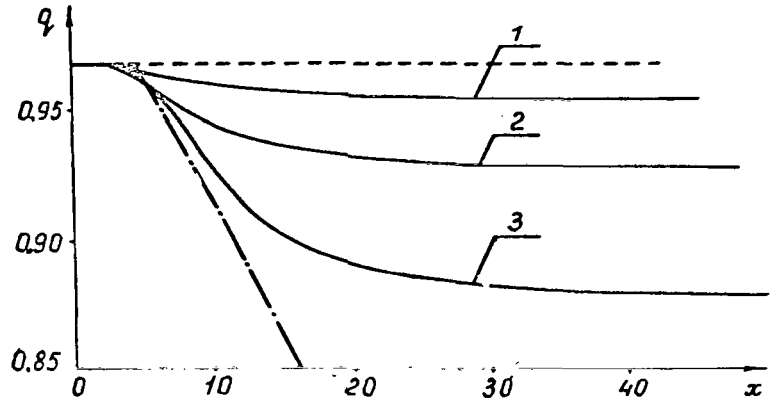

Figure 2.10.

Next we present examples of computations by the method of characteristics of external problems - i.e., problems of supersonic flow past bodies. P. I. Chushkin and N. P. Shulishnina [21] carried out a number of computations for axisymmetric flows of ideal air ($\varkappa = 1.4$) about blunted cones, with different apex half- angles ω, blunt nose shapes, and M_∞, the Mach numbers of the incoming flow. Calculations by the method of characteristics were carried out from some ray in the supersonic region, at which the initial data were taken from tables of O. M. Belotserkovskiy's numerical solution [22], obtained by the method of integral relations. Some results of these and other calculations are presented below.

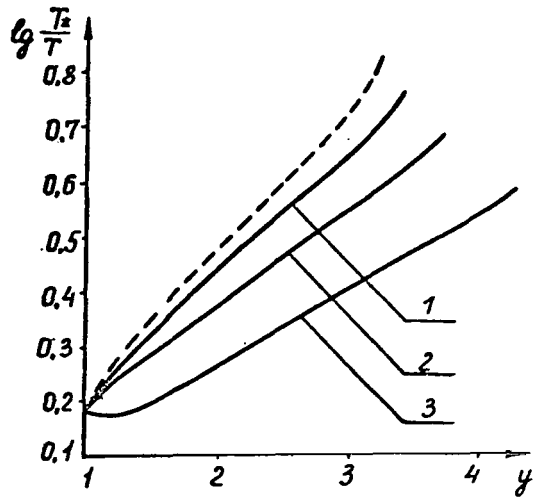

Figure 2.11.

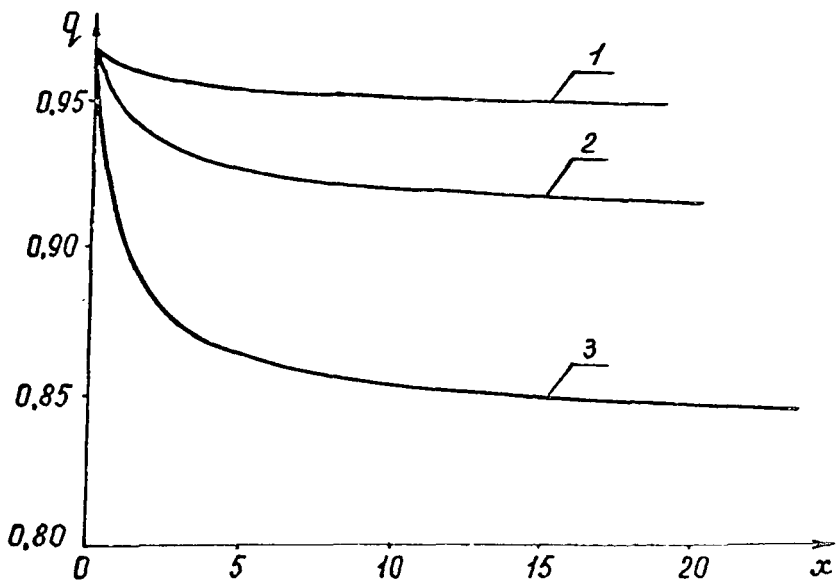

Figure 2.12.

Figure 2.13 shows the pressure distribution at the surface of blunted wedges (dashed lines) and cones (continuous lines) with different values of ω at $M_\infty = 4$. The pressure here is referred to p_0, the stagnation pressure, and the distance to the stagnation point along the x axis of the body is referred to R, the blunt-nose radius. The left branches of the curves correspond to blunted shapes. The graph

also shows horizontal dash-dot lines, which denote the pressures of corresponding nonblunt edges and cones.

/132

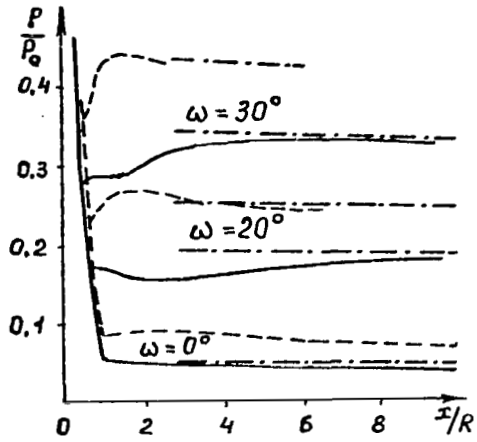

Figure 2.13.

--- - wedge; —— - cone.

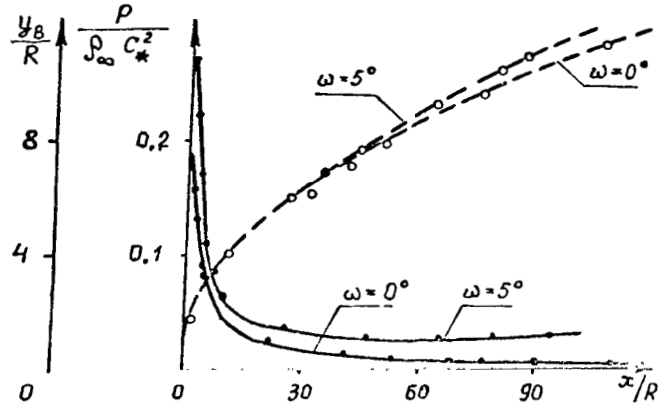

Figure 2.14.

The flow past a cone with $\omega = 5^{\circ}$ and cylinder with $\omega = 0^{\circ}$ with spherically blunted noses at $M_{\infty} = \infty$ and $\varkappa = 1.4$ was also calculated by the method of characteristics by strips [14]. The results of these calculations are depicted in Fig. 2.14, /133 which shows the pressure distribution at the body (continuous lines) and the shape of shock wave (dashed lines). These graphs also contain points dentoing corresponding values calculated by the ordinary method of characteristics.

I. N. Naumova also used the method of characteristics for calculating equilibrium flow of air past blunt cones. The blunting consisted of a sphere, and the initial data were taken from [23]. Figures 2.15 and 2.16 show the distribution of temperature T and Mach number M at cones with angles $\omega = 20$ and 0° (cylinder) and -20° (reversed cone) for the case of $M_\infty = 6$, $p_\infty = 1$ atm, $T_\infty = 300^\circ K$.

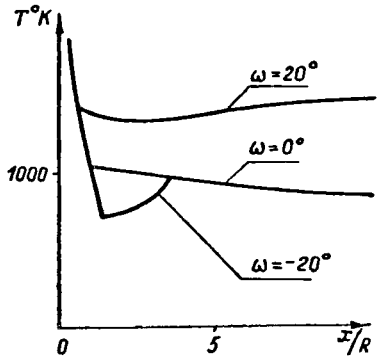

Figure 2.15.

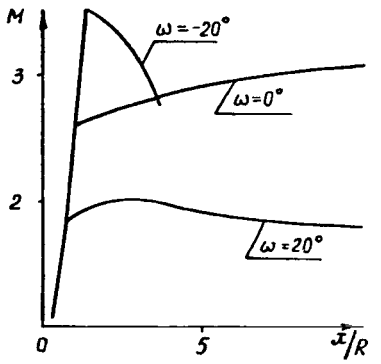

Figure 2.16.

3. Three-Dimensional Characteristics

I shall present here briefly only the idea of numerical schemes which quite recently produced interesting results in calculation of three-dimensional gas flows. Here one has the purely characteristic approach and the so-called "quasicharacteristic" method. The latter is so called because in it the functions are first estimated with respect to one of the variables (angular); this yields an approximate two-dimensional system of equations which is then solved by the ordinary method of characteristics simultaneously over all the interpolation planes. The work considered here was carried out by K. M. Magomedov [17] (first approach) and O. N. Katskova and P. I. Chushkin (second approach). /13

1. The three-dimensional method of characteristics

The main difficulties in using schemes of the three-dimensional method of characteristics is the absence of a unique interpretation of the characteristic relations. We now consider the principal tenets of the three-dimensional method of characteristics.

Following [17], we use as the sought functions the pressure p, entropy S and angle β and γ of the velocity vector $\vec{V} = V\{\cos\beta,\ \sin\beta\cos\gamma,\ \sin\beta\sin\gamma\}$ in the cylindrical coordinate system z, r, φ. Then the equation of motion of a steady inviscid

thermally nonconducting gas flow can be written in the form

$$\vec{k}_2\nabla\beta+\sin\beta\,\vec{k}_3\nabla\gamma+\frac{M^2-1}{\rho V^2}\,\vec{k}_1\nabla p=-\frac{\sin\beta\cos\gamma}{r}\,; \tag{2.10}$$

$$\vec{k}_1\nabla\beta+\frac{1}{\rho V^2}\,\vec{k}_2\nabla p=0; \tag{2.11}$$

$$\sin\beta\,\vec{k}_1\nabla\gamma+\frac{1}{\rho V^2}\,\vec{k}_3\nabla p=-\frac{\sin^2\beta\sin\gamma}{r}\,; \tag{2.12}$$

$$\vec{k}_1\nabla S=0, \tag{2.13}$$

where ρ, V^2 and M are functions of p and S; $\vec{k}_i\nabla f=\vec{k}_i\,\mathrm{grad}\,f=df/ds_i$ is the derivative of f along $\vec{k}_i$, while the unit vectors $\vec{k}_i$ (i = 1, 2, 3) form the local Cartesian coordinate system

$$\vec{k}_1=\{\cos\beta,\ \sin\beta\cos\gamma,\ \sin\beta\sin\gamma\};$$

$$\vec{k}_2=\{-\sin\beta,\ \cos\beta\cos\gamma,\ \cos\beta\sin\gamma\};$$

$$\vec{k}_3=\{0,\ -\sin\gamma,\ \cos\gamma\}.$$

We now reduce Eq. (2.10)-(2.13) to the characteristic form. This system contains derivatives in three directions $\vec{k}_i$ (i = 1, 2, 3). We now ask whether it is possible to reduce the number of directions along which derivatives are taken by some linear combination of Eqs. (2.10)-(2.13). The answer to this question has immediate pertinence to finding of characteristics (2.10)-(2.13), since some linear combination of starting equations at the characteristic surfaces does not contain derivatives which go beyond this surface. /135

We now consider Eqs. (2.10)-(2.13) at some fixed point of the flow. Multiplying each of them by $\omega_j\left(\sum_{j=1}^{4}\omega_j^2\neq 0\right)$ and summing, we get expressions such as

$$\vec{\Omega}_1\nabla\beta+\vec{\Omega}_2\nabla\gamma+\vec{\Omega}_3\nabla p+\vec{\Omega}_4\nabla S=F_1.$$

The condition that all the $\vec{\Omega}$ lie in the same plane, which is tangent to the characteristic surface with normal $\vec{n}$, is written as

$$\vec{n}\vec{\Omega}_j=0 \qquad (j=1, 2, 3, 4).$$

The necessary and sufficient condition for existence of a nontrivial solution of this

homogeneous system of linear equations in ω_j (zero equality of the determinant) has the form

$$n_1^2(M^2 n_1^2 - 1) = 0, \tag{2.14}$$

where $n_i = \vec{n}\vec{k}_i$ $(i = 1, 2, 3)$. The surface a normal to which satisfies this condition is a characteristic surface since, according to Eq. (2.14) there exists at least one linear combination of Eqs. (2.10)-(2.13) which does not contain derivatives going beyond this surface.

Equation (2.14) yields two types of characteristics: 1st root: $n_1 = 0$, along the streamline; all the planes passing through vector $\vec{V}$ are characteristic planes ($n_1 = \vec{n}\vec{V} = 0$ defines all the directions perpendicular to the streamline); 2nd root: $n_1 = \pm\frac{1}{M} = \pm\sin\alpha$, along the bicharacteristics forming a conoid (Mach cone) in the supersonic case ($\vec{n}\vec{V} = \pm V\sin\alpha$ defines the cone of characteristic normals).

We note that condition $n_1 = \sin\alpha$ is satisfied by the family of characteristic normals forming the cone of normals

$$\vec{n} = \sin\alpha\vec{k}_1 + \cos\alpha(-\sin t\vec{k}_2 + \cos t\vec{k}_3),$$

where t ranges from 0 to 2π.

The corresponding bicharacteristic directions /136

$$\vec{\tau} = \cos\alpha\vec{k}_1 + \sin\alpha(\sin t\vec{k}_2 - \cos t\vec{k}_3)$$

form the Mach cone. The solution of Eq. (2.14) for such $\vec{n}$ has the form

$$\omega_1 = \sin\alpha; \quad \omega_2 = \cos\alpha\sin t; \quad \omega_3 = -\cos\alpha\cos t; \quad \omega_4 = 0.$$

Linear combination of Eqs. (2.10)-(2.13) with these multipliers yields the compatibility condition along the wave bicharacteristic $\vec{\tau}(t)$

$$\sin t\frac{d\beta}{ds} + \frac{\sqrt{M^2-1}}{\rho V^2}\frac{dp}{ds} - \sin\beta\cos t\frac{d\gamma}{ds} + \frac{F}{r} = -\sin\alpha\left(\cos t\frac{d\beta}{d\sigma} + \sin\beta\sin t\frac{d\gamma}{d\sigma}\right), \tag{2.15}$$

where $\frac{df}{d\sigma}$ is the derivative along $\vec{\sigma}$, while df/ds is the derivative along $\vec{\tau}$. Vector $\vec{\sigma}(t) = \cos t\vec{k}_2 + \sin t\vec{k}_3$ is perpendicular to velocity vector $\vec{V}$ and to bicharacteristic $\vec{\tau}(t)$. If we select a plane containing $\vec{\tau}$ and $\vec{\sigma}$, then Eq. (2.15) can formally be treated as a characteristic relation (compatibility condition) at the characteristic surface tangent to this plane.

Condition $n_1 = 0$ is satisfied by the family of characteristic normals $\vec{n}(t) = \sin t\vec{k}_2 - \cos t\vec{k}_3$, $t \in [0, 2\pi]$. The corresponding solution of Eq. (2.14)

$$\omega_1 = 0; \quad \omega_2 = \cos t; \quad \omega_3 = \sin t; \quad \omega_4 = 0$$

makes it possible to obtain the compatibility condition along streamline $\vec{k}_1$

$$\cos t \frac{d\beta}{ds_0} + \sin\beta \sin t \frac{d\gamma}{ds_0} + \frac{\sin^2\beta \operatorname{sib}\gamma}{r} \sin t = -\frac{1}{\rho V^2}\frac{dp}{d\sigma}. \tag{2.16}$$

The constant-entropy condition

$$\frac{dS}{ds_0} = 0. \tag{2.17}$$

is also satisfied along the streamline ($\omega_1 = \omega_2 = \omega_3 = 0$, $\omega_4 = 1$).

Thus, the starting system of equations (2.10)-(2.13) has thus been reduced to the characteristic form (2.15)-(2.17). An infinite set of bicharacteristics passes through each point of the supersonic flow and compability conditions (2.15) /137
are satisfied along each of the former (along streamlines (2.16) and (2.17)). However, it should be remembered that the number of independent conditions cannot exceed the number of starting equations (i.e., four).

Thus, the compability conditions for three-dimensional steady flows contain internal derivatives along the nonbicharacteristic direction $\vec{\sigma}(t)$.

Some interpretation and the corresponding difference form of Eqs. (2.15)-(2.17) serve as a basis for selection of a given numerical method. The large number of schemes is due to selection of t and different forms in which Eqs. (2.15) and (2.16) are written, which can then be differently interpreted.

We consider the two-dimensional scheme. In the axisymmetrical (or plane) case $\gamma = 0$ and, if we take $t = \pi/2$ and $t = 3/2\,\pi$, then Eq. (2.15) does not contain its right-hand sides (Eq. (2.16) is identical to zero) and we shall have known compatibility conditions along characteristics of families I and II. Together with Eq. (2.17) we get a total characteristic system and unique interpretation, i.e., ordinary differential equations along the characteristic curves; here the compatibility

condition depends only on its characteristic directions. Here there are no difficulties in selecting the numerical scheme, since Massau's classical scheme is used anyway.

While condition $\vec{k}_1 \vec{n} = \sin\alpha$ (or $\vec{k}_1 \vec{n} = 0$) in the two dimensional case defines only the characteristic curves, in the three-dimensional case the characteristics may be of a large variety, such as a characteristic surface, conoid, bicharacteristics, or, in a given point, respectively, planes, Mach cone and the latter's generatrices.

We now consider Eq. (2.15) for three values of $t \in [0, 2\pi]$ ($t_1 < t_2 < t_3$). We draw through the given point A of the supersonic flow the Mach cone ($\vec{r} = \vec{r}_A + s\vec{\tau}(t)$, s and t being the coordinates of the cone's surface) and three planes tangent to the cone (Fig. 2.17). Tangency will exist, by definition, along the generatrices of cone A_i, i.e., along the bicharacteristics $\vec{\tau}(t)$. One can consider two schemes with tetrahedron $AB_1B_2B_3$.

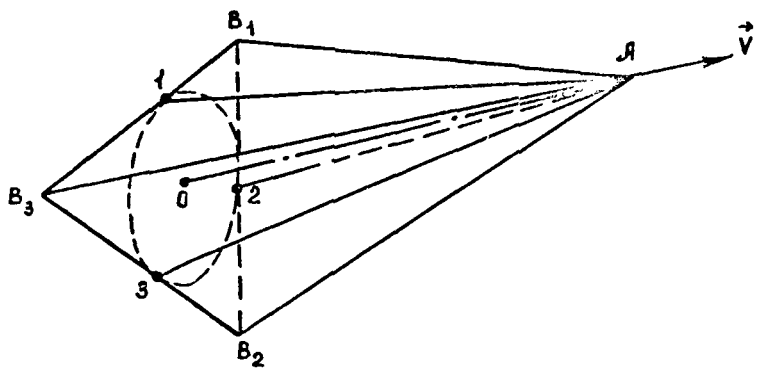

Figure 2.17.

1. The direct tetrahedral scheme. Let all the variables in point $B_1B_2B_3$ be known. Knowing directions $\vec{\sigma}(t_i)$ for example, B_{ij} (i, j = 1, 2, 3), we can find t_i and the characteristic normals $n(t_i)$ of planes AB_iB_j. At the intersection of the corresponding planes we find the sought point A, the variables at which are determined from Eqs. (2.15) in difference form at t_1, t_2 and t_3 and from Eq. (2.17), expressing the conservation of entropy along the streamline. This scheme was developed by V.V. Rusanov in 1953 [15], and was recently used by Yu. N. Podladchikov [16]. /138

2. The inverse tetrahedral scheme. Let the coordinates of computed point A be known. Selecting quite arbitrary values of t_i (i = 1, 2, 3) (but in such a manner that they would uniformly extend over segment $[0, 2\pi]$) we drop characteristic

planes onto the closest surface where the solution is known. All the hydrodynamic function in point A are found by determining the variables and derivatives along direction $\vec{\sigma}(t_i)$ from data at this surface and by writing Eq. (2.15) in difference form (derivatives with respect to $\vec{\sigma}$ can be determined from polynomials of through approximation on the plane, as this is done, for example in [20]).

Other schemes, based on a different interpretation of Eq. (2.15) are possible. However, in all these schemes the derivatives at nonbicharacteristic directions $\vec{\sigma}$ are replaced by finite-difference relationships. This raises many questions of the proper allowance for the region of influence, on the advantage of these methods over ordinary net-point methods, etc.

3. The characteristic scheme. To provide a more natural interpretation of Eq. (2.15), we shall first prove the following assertion. Starting equations (2.10)-(2.13) for $M > 1$ at a given point can be transformed in such a manner that the system thus obtained will contain derivatives only along bicharacteristic directions. /139

In fact, let us consider the bicharacteristics at $t = 0$, π:

$$\vec{\tau}_i = \cos\alpha\vec{k}_1 + (-1)^i \sin\alpha\vec{k}_3 \qquad (i = 3, 4).$$

Then

$$\vec{k}_3 = \frac{\vec{\tau}_4 - \vec{\tau}_3}{2\sin\alpha}; \quad \frac{df}{d\sigma} = \vec{k}_3\nabla f = \frac{1}{2\sin\alpha}\left(\frac{df}{ds_4} - \frac{df}{ds_3}\right).$$

Taking Eq. (2.15) at $t = \frac{\pi}{2}$ and $t = \frac{3}{2}\pi$, Eq. (2.16) at $t = \frac{\pi}{2}$, as well as Eq. (2.17), and substituting the values of $\frac{df}{d\sigma}$ into these equations, we get a system which is equivalent to the starting system (2.10)-(2.13) and which contains derivatives only in the bicharacteristic directions $\vec{k}_1$ and $\vec{\tau}_j$, $j = 0, 1, 2, 3, 4$, $\left(\vec{\tau}_3(0), \vec{\tau}_4(\pi), \vec{\tau}_1\left(\frac{\pi}{2}\right), \vec{\tau}_2\left(\frac{3}{2}\pi\right)\right)$:

$$\left.\begin{aligned} &\frac{d\beta}{ds_i} + \frac{(-1)^i}{\rho V^2 \operatorname{tg}\alpha}\frac{dp}{ds_i} = -(-1)^i \sin\beta\sin\alpha\left[\frac{\cos\gamma}{r} + \frac{1}{2\sin\alpha}\left(\frac{d\gamma}{ds_4} - \frac{d\gamma}{ds_3}\right)\right]; \\ &\sin\beta\frac{d\gamma}{ds_0} = -\frac{\sin^2\beta\sin\gamma}{r} - \frac{1}{\rho V^2 2\sin\alpha}\left(\frac{dp}{ds_4} - \frac{dp}{ds_3}\right); \quad \frac{dS}{ds_0} = 0. \end{aligned}\right\} \qquad (2.18)$$

In light of the above we shall now present a new interpretation of characteristic relations (the K. M. Magomedov scheme [17]). We consider a given point A,

located at small distance h downstream of a three-dimensional type surface, at which the variables are known (Fig. 2.18). Replacing the derivatives in Eq. (2.18) by equations "backwards" along bicharacteristics A_i($i = 1, 2, 3, 4$) and streamline AO, we get an explicit difference scheme approximating the initial differential equations to with $0(h^2)$ (scheme of first-order accuracy). To obtain second-order accuracy we could have used points of a new plane P_0, the former being close to point A, but this makes the scheme implicit and has little in common with the method of characteristics. /140

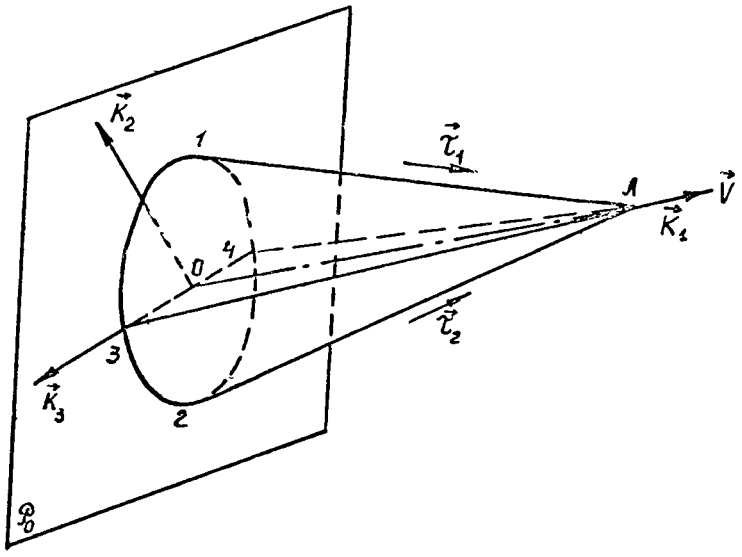

Figure 2.18.

K.M. Magomedov showed that in order to calculate a point at a shock wave and at a free surface it is necessary and sufficient to make use of only a single combination of gasdynamic equations, containing derivatives passing beyond the surface being considered, for example, a characteristic equation only along a single bicharacteristic. The numerical algorithm of calculating boundary points in this scheme, as in the two-dimensional method, markedly simplifies the scheme and the only unknown is determined by Newton's method.

The method's stability is ensured by the fact that the variables at points of intersection of characteristics with the preceding plane are determined by interpolation nodes, lying outside the domain of influence shown in Fig. 2.18. Actual calculations showed that linear interpolation should be used.

The programming logic is sufficiently simple and the computations are similar to those for the two-dimensional problem. The principal difference of this method is the need of interpolation from two variables at the preceding plane. /141

In conclusion we consider a question pertaining to construction of a specific computational scheme. Obviously the selection of dependent variables is of great importance in any numerical method. Introduction of angles β and γ at the start was found inconvenient in [17], since this produces a singularity as $\beta \to 0$. Replacement of the direct components of the velocity vector, pressure, density and entropy contained in the starting equations with tangents of the two angles of the velocity vector, pressure and enthalpy reduces the number of unknowns, simplifies the arithmetical expressions and is a natural generalization of two-dimensional variables. In addition, these variables are most convenient when allowance is made for equilibrium physico-chemical transformations.

Using the three-dimensional method of characteristics and employing this scheme, K. M. Magomedov calculated three-dimensional flows around bodies of different shape (sharp and blunt bodies of revolution at different angles of attack α, delta wing with blunted edges, etc.). Some results of these calculations are presented here (Figs. 2.19-2.23).

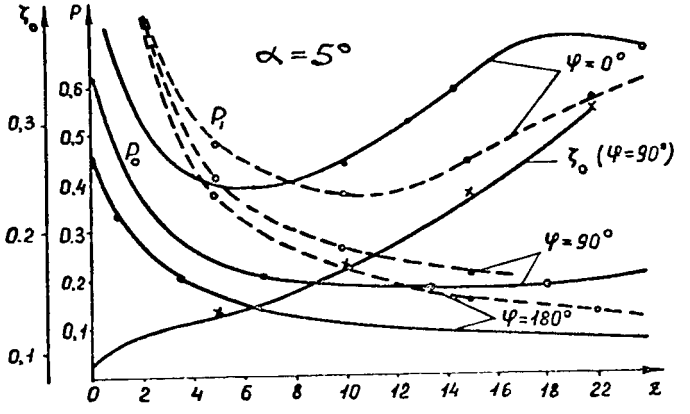

Figure 2.19.

Figure 2.19 shows for the case $M_\infty = \infty$, $\varkappa = 1.4$, $\alpha = 5^\circ$, $\omega = 9^\circ 30'$ the pressure distribution at blunted circular cones p_0 (continuous lines) and p_1 (dashed lines) along three generatrices $\varphi = 0$, 90 and 180°, as well as the parameter of the circumferential velocity component $\zeta_0 = \frac{w}{\sqrt{u^2+v^2}}$. This figure also shows data obtained by Yu. N. D'yakonov [24] by the net-point method suggested by K.I. Babenko and G. P. Voskresenskiy [25] (points and crosses). We note that the variables used in practical calculations were η and ζ, differing from those considered in [17], that is $\eta = \frac{V}{u}$, $\zeta = \frac{w}{\sqrt{u^2+v^2}}$.

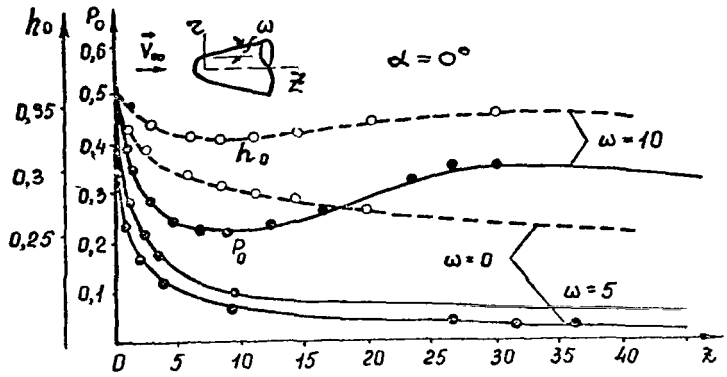

Figure 2.20.

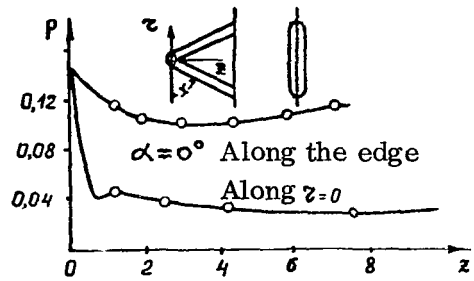

Figure 2.21.

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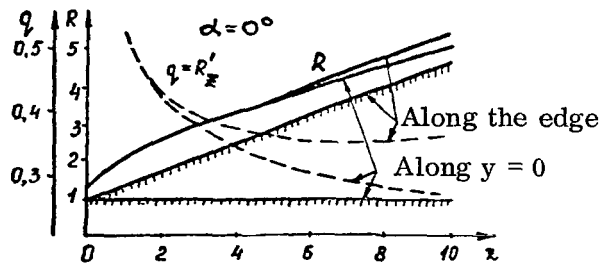

Figure 2.22.

Calculations were also made for hypersonic air flows past cones with allowance for equilibrium physico-chemical transformations. Figure 2.20 compares the applicable data with results obtained by V. V. Lunev and V. G. Pavlov (V_∞ = 7500 m/sec, M_∞ = 23.5, $\alpha = 0^0$) by the ordinary two-dimensional method of characteristics (points).

Some results of numerical solution of the more complicated three-dimensional problem, i.e., flow past a delta wing with cylindrically blunted edges and with a spherical nose, are shown in Figs. 2.21-2.22 ($M_\infty = 6$, $\varkappa = 1.4$, $\chi = 70^\circ$, $\alpha = 0^\circ$). /142

The pressure along the wing's lines of symmetry is shown in Fig. 2.21. It is seen from Fig. 2.22 (where the shock-wave shape is shown) that the shock wave remains axisymmetrical at a sufficiently large distance from the nose. The complicated flow pattern is due to interaction of the axisymmetrical flow about the spherical nose and the flow emanating from edges at the flat part of the wing.

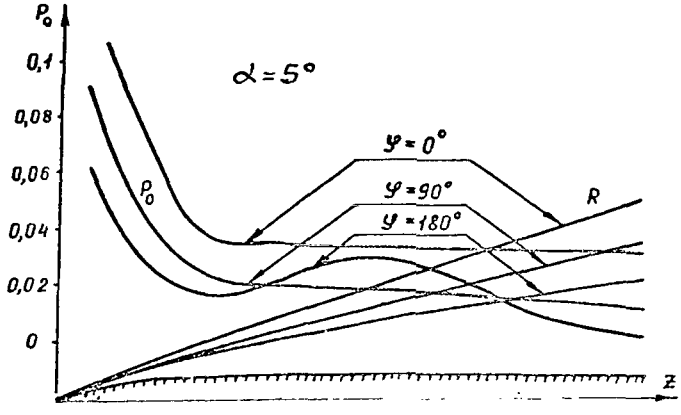

Figure 2.23.

Figure 2.23 shows the distribution of pressure p_0 and the cross section R of the shock wave by meridional planes $\varphi = 0$, 90 and 180° at an elongated cylinder with a sharp nose ($\varkappa = 1{,}4$) with $M_\infty = 5$ at angle of attack $\alpha = 5^\circ$. As is seen from this graph, the pressure minimum at elongated bodies for specific values of x is found at $\varphi < 180^\circ$. A similar pattern exists also at the cylinder with a spherical nose. /144

2. The "quasicharacteristic" method

In conclusion we shall describe the idea underlying the construction of the "quasicharacteristic" method developed by O. N. Katskova and P. I. Chushkin [18], and shall present some results of calculations of three-dimensional flows.

Flows past axisymmetric (or similarly shaped) bodies are most conveniently considered in the cylindrical coordinate system (r, x, ψ). We "straighten out" the region between the body and the shock wave, by replacing r with a new independent variable $\xi = \frac{r - r_T}{r_B - r_T}$ (Fig. 2.24). The starting system of equations is then written in the variables (ξ, x, ψ). /145

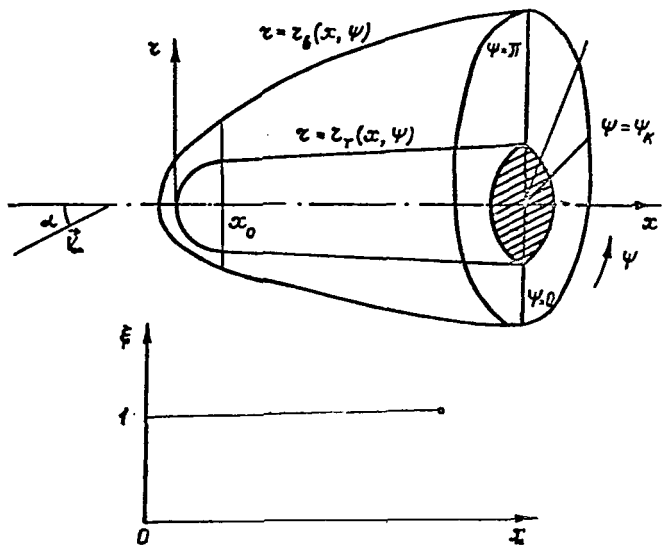

Figure 2.24.

The three-dimensional system of differential equations is now reduced to its approximating two-dimensional system, by eliminating derivatives with respect to ψ. For this, in region $0 \leq \psi \leq \pi$ we draw $l + 1$ meridional planes located at equal distances ψ from one another $\psi = \psi_k = \frac{k\pi}{l}$ $(k = 0, 1, \ldots, l)$. Then functions under the derivative sign are represented by interpolational trigonometric polynomials, using the above meridional planes as interpolation points.

For odd surfaces we have

$$F(x, \xi, \psi) = \sum_{k=1}^{l-1} a_k(x, \xi) \sin k\xi ,$$

for even surfaces

$$\tilde{F}(x, \xi, \psi) = \sum_{k=0}^{l} b_k(x, \xi) \cos k\xi ,$$

where the values of a_k and b_k depend on the values of the approximating functions at the meridional planes. From these representations we get expressions for derivatives $\left(\frac{\partial F}{\partial \psi}\right)_k$ and $\left(\frac{\partial \tilde{F}}{\partial \psi}\right)_k$.

Substituting into the starting system of equations, we get an approximating system of partial differential equations in x and ξ. The dependent variables will here be the values of the sought functions at planes $\psi = \psi_k$.

The above system has two families of real characteristics. Still another family of curves, which can be conditionally called "streamlines" (by analogy with streamlines of axisymmetric flow) will also have characteristic properties.

Using the scheme of the two-dimensional method of characteristics "by strips" $x = \text{const}$, the approximating system is solved numerically at a number of points of intersection $\xi = \text{const}$ simultaneously in all the meridional planes $\psi = \psi_k$.

The computational scheme is as follows: a) characteristics and "streamlines" are generated from the nodes onto the preceding strip. The values of the functions at the points of intersection at the preceding strip are determined by quadratic interpolation. The characteristic equations are written in difference form. b) /146
Calculations are started from the shock wave $\xi = 1$, where equations of characteristics of the first family and relationships at the shock wave are used. Then internal points of the field are considered, and finally, points on the body $\xi = 0$, where equations of characteristics of the second family and the impermeability condition are used.

The implicit scheme thus constructed has the important advantage of being of the second order of accuracy with respect to x (all the remaining schemes have the first order of accuracy). This makes possible savings of machine time by selecting a coarser step along x for the specified accuracy. In addition, this scheme has less rigorous limitations, which are determined by the stability criterion.

The method of integral relations can also be used for calculation of three-dimensional flows. For this: 1) the starting system is written in divergence form; 2) variables (r, x, ψ) are replaced by variables (ξ, x, ψ); 3) trigonometric interpolation is carried out for ψ; 4) the ordinary method of integral relations is used at planes $\psi = \psi_k$ (interpolation for ξ, which yields the Cauchy problem for the approximating system with respect to x).

We shall now present some results obtained by the "quasicharacteristic" method. The examples considered were different cases of supersonic flow past blunt cones at an angle of attack in the flow of ideal air ($\varkappa = 1.4$). The cones had circular cross sections, and the blunt noses were spherical. The cases differed in the apex half-angle of the cone ω, incident flow Mach number M_∞ and angle of attack α.

Some of the results for three-dimensional supersonic flow past a blunt cone are presented in Figs. 2.25-2.29 obtained for $M_\infty = \infty$, $\omega = 5^\circ$, $\alpha = 10^\circ$. The linear dimensions in all the graphs were referred to the blunting radius, and the origin of the cylindrical coordinate system is located in the forward point of the body.

The individual versions were calculated with a different computational grid. The number n (the number of points in each meridional plane) was taken as 25. It is /147
shown by comparison with corresponding calculations using $n = 50$ that the value of n used ensures sufficient accuracy. The number l of meridional surfaces was also different in the several computations. All the principal cases were calculated for five meridional planes ($l = 4$, shown by solid lines on the graph), while some were computed with three meridional surfaces ($l = 2$, shown by "circles" on the graphs).

Figure 2.25 shows the shape of the shock wave in sections ψ = const, while Fig. 2.26 depicts the pressure distribution along the body's surface (on rectilinear segment of the generatrix). The pressure variation between the shock wave and the body's surface along strips x = const for $\psi = 0$ and $\psi = \pi$ is given in Figs. 2.27-2.28. /148

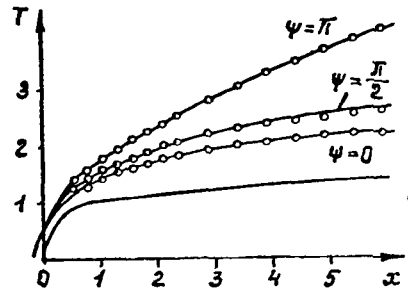

Figure 2.25.

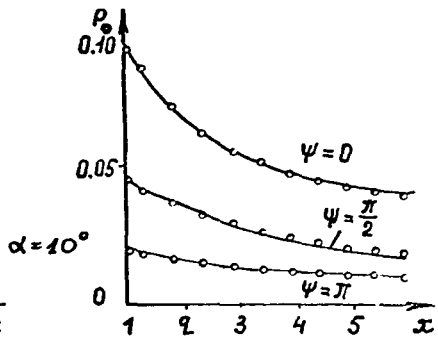

Figure 2.26.

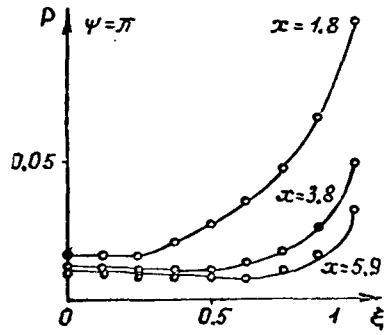

Figure 2.27.

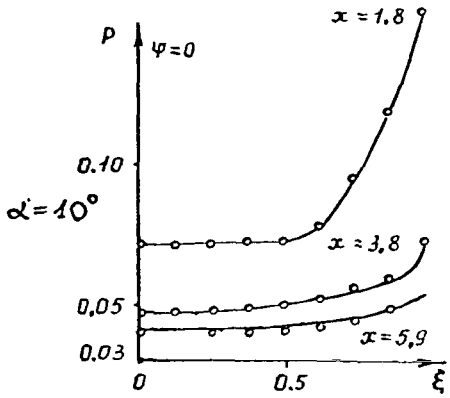

Figure 2.28.

Figure 2.29 compares the pressure distribution at a cone with $\omega = 20^0$ for $M_\infty = \infty$ and $\alpha = 10^0$ with the corresponding results (but at $M_\infty = 20$) obtained by Yu. N. D'yakonov [25], who used the Babenko-Voskresenskiy finite-difference method [25], shown by dashed curves on the graph.

Finally Figs. 2.30-2.31 show computations for the flow past and through bodies at angle of attack α under conditions when the shock wave is attached to the forward edge. Figure 2.30 shows the shock-wave wakes in the plane of symmetry of

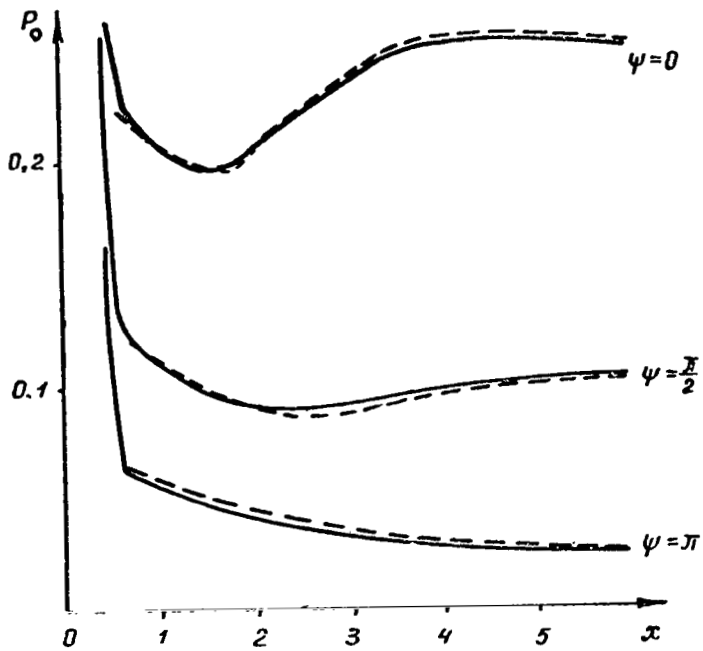

Figure 2.29.

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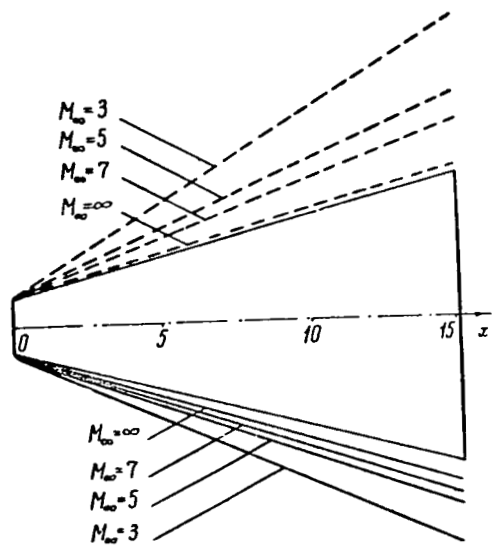


Figure 2.30.

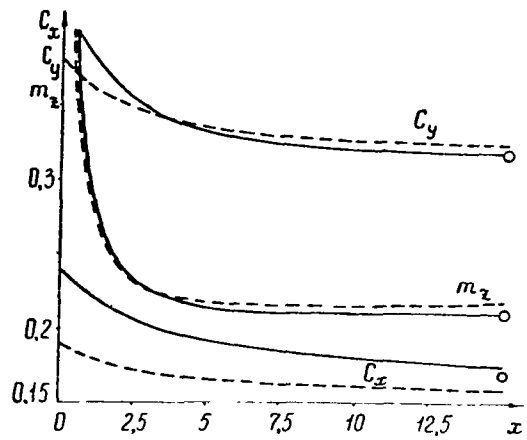


Figure 2.31.

flow for a body with internal flow with a taper angle of 15° for $\alpha = 10^\circ$ and $M_\infty = 3, 5, 7$ and ∞. Figure 2.31 depicts the variations at the outer surface of the same body in the aerodynamic coefficients C_x, C_y and m_z (the continuous curve is for $M_\infty = 5$, the dashed curve is for $M_\infty = \infty$). /149

PART III. CALCULATION OF THE FLOW AROUND BLUNTED BODIES WITH A DETACHED SHOCK WAVE /150

Above we considered the ideas underlying construction of numerical algorithms using the method of integral relations and examined specific schemes for the method of characteristics. Now we shall demonstrate different approaches to construction of steady-state numerical schemes for computation of the flow field in the problem of supersonic flow past blunt bodies. In spite of the fact that presently many algorithms are available for solving the mixed problem in the region of influence of the blunting, their development and refinement is continuing, since practical needs require the solution of increasingly new and complicated problems (flows with chemical reactions, radiation, viscous flows, flow past bodies with a sharp corner in the contour generatrix, etc.). Subsequently we shall consider in sufficient detail the use of the method of integral relations for this problem, and we shall also examine a scheme of the method of straight lines and an attempt to regulize an incorrect inverse problem of gasdynamics using as an illustration the construction of special difference systems.

Statement of the problem. Let a supersonic flow of an ideal inviscid gas flow onto a blunted body at constant velocity. The overall pattern of flow around a spherically blunted cone ($\omega = 20^\circ$, $M_\infty = 6$) is shown in Fig. 3.1. The region behind the detached shock wave (the location of which is not a priori known) consists of two zones: near the nose section of the body, in the region of influence of the blunting, bounded by limiting characteristic CE, the gas flow is mixed (ED is the sonic line, /151

where the equations change type). This region of influence is replaced further downstream by a supersonic zone (hyperbolic-type equations). The flow in the mixed region is computed independently of the supersonic zone and, conversely, the flow in the supersonic zone is determined, for example, by the method of characteristics, after the region of influence of the blunting has been computed.

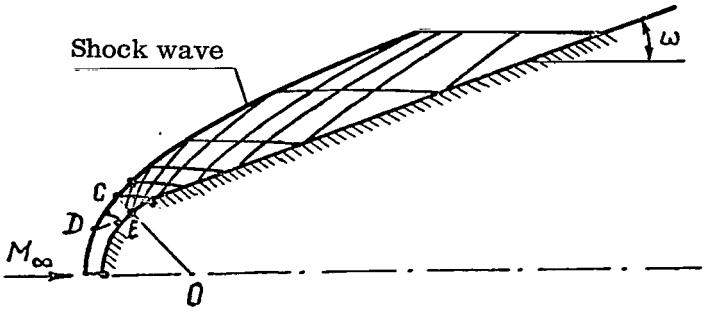


Figure 3.1.

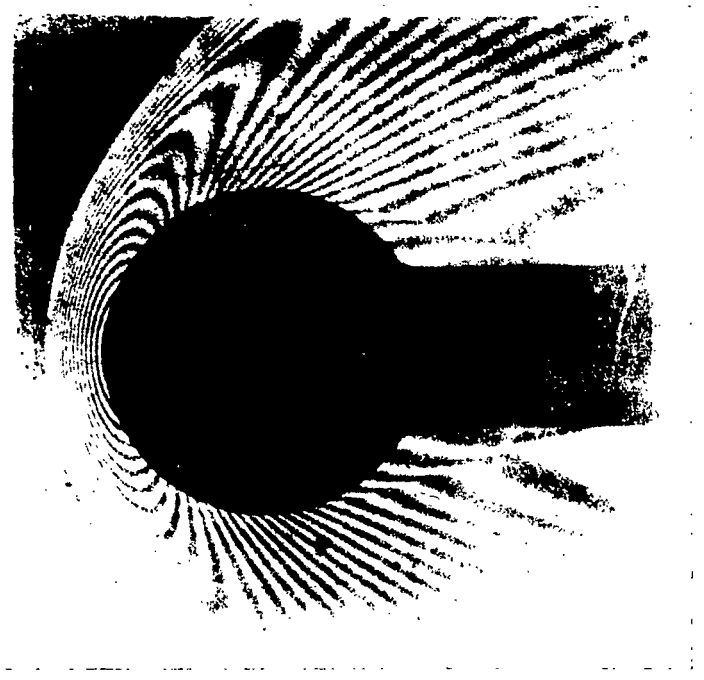


Figure 3.2.

There exist two fundamentally different types of numerical methods for calculating mixed gas flows between a detached shock wave and the nose section. In direct-type methods the shock wave shape is determined for a given body geometry.

In inverse type methods the shock wave shape is specified and the corresponding body geometry is found. Here one has to deal with an incorrectly stated Cauchy problem.

The determination of the flow field in the region of influence of the blunting is thus a complicated mathematical problem which is described by a nonlinear system of mixed-type partial differential equations, with a part of the boundary conditions specified at "moving" lines or surfaces. /153

The solution of the above problem is of great practical interest, since blunt aerodynamical shapes are extensively used in flights at hypersonic velocities. In spite of the fact that the blunting may be geometrically small as compared with the entire body, it has an extraordinarily important effect on the entire flow pattern, forming the flow behavior being considered here. /152

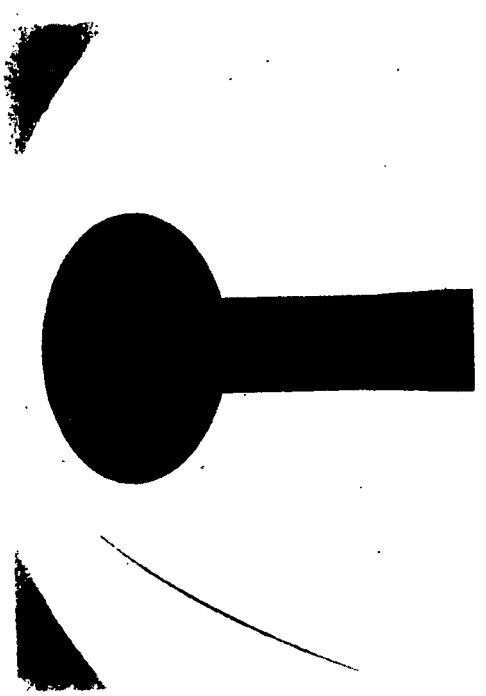


Figure 3.3

Figures 3.2-3.4 present photographs of flows around bodies of different shapes in a wind tunnel obtained by G.M. Ryabinkov [26]. Figure 3.2 shows an interferogram of the flow about a sphere at $M_\infty = 3$ (where constant-density lines are clearly seen). The photograph of Fig. 3.3 was obtained for an ellipsoid of

revolution with $\delta = 1.5$ at $M_\infty = 3$ by the shadow method, while that of Fig. 3.4 was obtained for $\varepsilon = 0.5$ and $M_\infty = 3$ using Töpler's instrument. At the end of the ensuing section we shall compare the results computed by the method of integral relations with G. M. Ryabinkov's experimental data.

Figure 3.4.

We shall consider algorithms for solving this problem obtained by the method of integral relations and the method of straight lines, and we shall also describe a finite-difference scheme for a system of equations with pseudo-viscosity.

1. Algorithms of Numerical Schemes of the Method of Integral Relations for Computation of Mixed Gas Flows

Nomenclature

s - arc length measured along the body's contour (s = 0 at the stagnation point);
n - normal to the body's surface;
R - radius of curvature of the body;
θ, σ - angle formed by the tangent to the body's contour and the shock wave, respectively, and the direction of incoming flow;
r - normal distance from the body's axis of symmetry;
ε - normal distance from the body's surface n = 0 to the shock wave $n = \varepsilon(s)$;
$s = s_1(\xi)$ - equation of the boundary characteristic ($\xi = n/\varepsilon(s)$); /154
$\mathfrak{R}$ - universal gas constant;
m_∞ - molecular weight of gas in incoming flow;

u, v - components of velocity w in directions n and s, respectively, referred to the maximum velocity w_{max};

w_t, w_n - components of velocity w, respectively tangential and normal to the shock wave, referred to w_{max};

c - speed of sound referred to w_{max};

M - Mach number;

ρ - density, referred to the gas density of incoming flow ρ_∞;

p - pressure, referred to $\rho_\infty w_{max}^2$;

T - temperature referred to $w_{max}^2 m_\infty / \Re$;

S - entropy referred to $\Re / 2 m_\infty$;

C_i - mass concentration of the ith component;

h_i, h - specific enthalpy of the ith component and of the entire mixture, respectively, referred to stagnation enthalpy $h_{00} = w_{max}^2/2$;

m_i - molecular weight of the ith component

ω_i - mass rate of formation of the ith component;

μ, λ - number of gas-mixture components and of different kinds of atoms in the incoming flow, respectively;

$\varkappa$ - ratio of specific heats.

The linear dimensions are referred to the radius of curvature of the body at the stagnation point R_0; subscript ∞ denotes quantities up to the shock wave; $\nu = 0$ or 1, respectively, for the plane or axisymmetrical cases.

As was previously noted, the problem of supersonic flow past bodies with detached shock wave involves consideration of mixed flow in the region of influence of the blunting. The method of integral relations [1, 2, 27] makes it possible to construct for such flows various formal numerical algorithms, efficient utilization of which makes possible computer calculation with the required accuracy of various cases of external flows. These algorithms can also be used in the study of mixed flows in nozzles, in solving certain geodetic problems, etc.

We consider the direct steady problem, with all the boundary conditions of the /15
starting system of equations satisfied in each approximation. The problem consists in determining a unique continuous solution with continuous derivatives (with the possible exception of boundary points) in the minimal region of influence of the blunting (Figs. 3.5-3.7) bounded by shock wave AB_3, axis of symmetry AE (in the two-dimensional case), the body's contour DE and the boundary (limiting) characteristic B_3D. Figures 3.5-3.7 show the flow patterns ($M_\infty = 4.0$, ideal gas $\varkappa = 1.4$), obtained by calculation for a circular cylinder (Fig. 3.5), sphere (3.6) and a sphere with a sharp corner (Fig. 3.7). The problem under study has regions of influence of the type shown in Fig. 3.8.

The boundary conditions for the starting system are specified at the shock front, the axis of symmetry and the body's surface. We thus obtain a problem with

an "open" upper boundary and the problem is made unique only by the condition of regularity, which singles out the class of solutions with a bounded derivative.

The use of exact integrals of the system, divergence form in writing the partial derivatives, as well as the approximate representation of integrals in this method make possible a more exact approximation of the starting differential operator of the system.

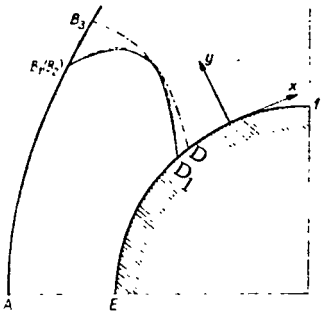


Figure 3.5.

$M_\infty = 4$, $\delta = 1$, $\nu = 0$, $\varkappa = 1.4$; —— - sonic line ($w = c$) - B_1D_1; --- - singular line ($w_x = c$) - B_2D_1; -.- - boundary characteristic B_3D.

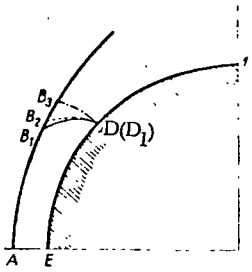


Figure 3.6.

$M_\infty = 4$, $\delta = 1$, $\nu = 1$, $\varkappa = 1.4$; —— - sonic line - B_1D_1; - - singular line ($w_x = c$) - B_2D_1; -.- - boundary characteristic B_3D.

The type of algorithm depends primarily only on the variable with respect to which the approximation is made. Allowance for the different properties of flows, application of various kinds of complete systems of base functions and the different structure of approximating grids do not affect the substance of the algorithm, and hence the computational program. From a universal program which can be used for calculating an arbitrary number N of the approximation, one can estimate the accuracy and rate of practical convergence of the method of integral relations.

Three types of function (approximating grid) representations and their associated three types of algorithms were found convenient for different external flow regimes and body shapes [28, 26].

Lines $n = (k/N) \cdot \varepsilon(s)$; $k = 1, 2, \ldots, N - 1$ were found convenient for use as approximation lines in the flow of gas at high M_∞ past bodies of complex configuration (scheme I, Fig. 3.9). This subdivision of the region of integration is expedient /156 if the flow variables vary sufficiently smoothly across the shock and the shock proper

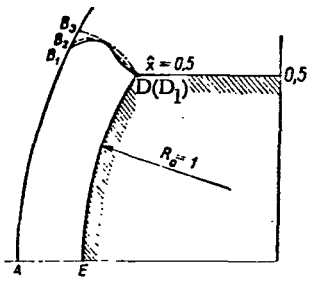


Figure 3.7.

$M_\infty = 4$, $\nu = 1$, $\varkappa = 1.4$ (sphere with sharp corner $\hat{x} = 0.5$); —— - sonic line B_1D_1 ($w = c$); --- - singular line ($w_x = c$) B_2D_1; -.- - boundary characteristic B_3D_1.

is sufficiently thin. In addition, in this case we get more complete information at the body and shock wave, which is important from the practical point of view.

Under external flow conditions in which it is necessary to consider the relaxation properties of the gas (nonequilibrium dissociation, ionization, etc.), for bodies with no abrupt changes in curvature it is expedient to use an algorithm involving the subdivision of the region by lines $s = (k/N) \cdot s_1(\xi)$; ($k = 1, 2, \ldots, N - 1$ (scheme II, Fig. 3.10). In a number of cases, for example, in computation of viscous flows, flows with comparatively low M_∞, it is convenient to use double approximation of functions, wherein the functions are first represented according to the first scheme, integration is carried out, and then approximation is carried out using the second scheme (scheme III in Fig. 3.11). As a result we get a system of nonlinear algebraic equations which is solved numerically. This scheme provides a substantial economy of machine time. /158

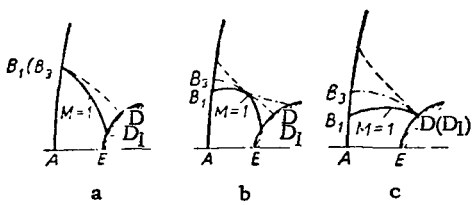


Figure 3.8.

--- - characteristic of family I;
-.- - characteristic of family II; B_1D_1 - sonic line; B_3D - boundary characteristic.

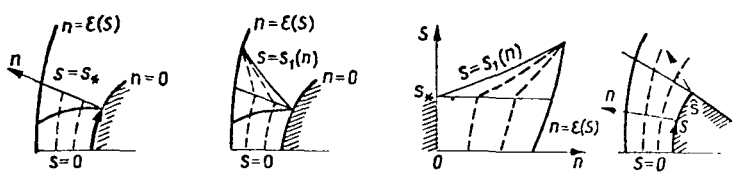


Figure 3.9.

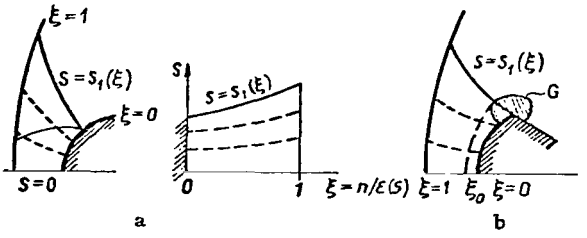


Figure 3.10.

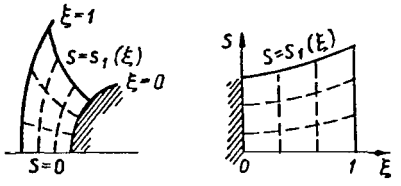


Figure 3.11.

1. Scheme I. The computational algorithm for this scheme (which was developed in 1957 [27]) will be constructed for supersonic flow past an axisymmetrical body with allowance for equilibrium dissociation of the gas. The independent variables here are the s, n orthogonal coordinate system (see Fig. 3.9). The complete system of equations (equations of motion projected on n, continuity, equation for the streamline, energy and the Bernoulli integral) in the dimensionless form is written as:

$$\left.\begin{aligned} &\frac{\partial \tilde{Z}}{\partial s}+\frac{\partial}{\partial n}A\tilde{H}=\tilde{Y}; \quad \frac{\partial \tilde{t}}{\partial s}+\frac{\partial}{\partial n}A\tilde{h}=0;\\ &\frac{\partial \psi}{\partial s}=\tilde{t}\,\frac{\partial n}{\partial s}-A\tilde{h}; \quad S=S_*(\psi); \quad w^2+h=1, \end{aligned}\right\} \tag{3.1}$$

where $S_*(\psi)$ is the value of the entropy at the shock wave.

The thermodynamic relationships have the form

$$TdS=dh-2\frac{dp}{\rho}; \quad h=h(T,p); \quad \rho(p,T)=\frac{p}{T}G(p,T), \tag{3.2}$$

Here

$$\tilde{Z}=rZ; \quad \tilde{h}=rh; \quad \tilde{t}=rt; \quad \tilde{H}=rH; \quad \tilde{g}=rg;$$
$$Z=\rho uv; \quad h=\rho u; \quad H=p+\rho u^2; \quad g=p+\rho v^2;$$
$$r=r_0+n\cos\theta; \quad A=1+n/R; \quad \tilde{Y}=\tilde{g}/R+AP\cos\theta.$$

We now write the boundary conditions of the problem. At the surface of the

body n = 0

$$u = 0; \quad \psi = 0; \quad S = S_*(0) = 0.$$

at the shock wave $n = \varepsilon(s)$

$$w_t = w_\infty \cos\sigma; \quad \rho w_n = w_\infty \sin\sigma;$$

$$p = p_\infty - \rho W_n^2 + (w_\infty \sin\sigma)^2; \quad \psi = \frac{r^2 w_\infty}{2};$$

$$\frac{dw_n}{d\sigma} = \frac{w_\infty h_T \cos\sigma - w_n(DE + \rho_T E_1)}{\rho h_T - w_n(Dw_\infty \sin\sigma + 2\rho_T w_n)};$$

$$\frac{dp}{d\sigma} = E - w_\infty \sin\sigma \frac{dw_n}{d\sigma};$$

$$\frac{dS}{d\sigma} = -\frac{2}{T}\left[w_n \frac{dw_n}{d\sigma} + \frac{1}{\rho}\frac{dp}{d\sigma} - \frac{E_1}{2}\right];$$

$$\frac{dT}{d\sigma} = \frac{1}{h_T}\left[E_1 - h_p \frac{dp}{d\sigma} - 2w_n \frac{dw_n}{d\sigma}\right].$$

/159

Here $E_1 = w_\infty^2 \sin 2\sigma$; $E = E_1 - W_n w_\infty \cos\sigma$; $D = \rho_p h_T - \rho_T h_p$. At the axis of symmetry $s = 0$, $v = 0$, $\psi = 0$, $S = 0$. The starting system (3.1)-(3.2) contains two partial differential equations, which are here represented approximately.

Functions $\tilde{Z}$, $\tilde{t}$, $\tilde{Y}$ are represented as follows:

$$\tilde{Z} = \sum_{i=1}^{N} Z_{(i)}(s)\xi^i; \quad \tilde{t} = \sum_{i=0}^{N} t_{(i)}(s)\xi^i; \quad \tilde{Y} = \sum_{i=0}^{N} Y_{(i)}(s)\xi^i; \quad \xi = n/\varepsilon(s).$$

The region of integration is subdivided into N strips by lines $\xi = k/N$ ($k = 1, 2, \ldots, N - 1$) and we integrate each of the partial differential equations, which are approximated, with respect to n from the zeroth to each of the N remaining lines

$$\int_0^{(k/N)\varepsilon(s)} \frac{\partial}{\partial s}\left[\sum_{i=1}^{N} Z_{(i)}\xi^i\right] dn + \int_0^{(k/N)\varepsilon(s)} \frac{\partial}{\partial n}(A\tilde{H})\, dn =$$

$$= \int_0^{(k/N)\varepsilon(s)} \sum_{i=0}^{N} Y_{(i)}\xi^i\, dn;$$

$$\frac{d}{ds}\left[\varepsilon(s)\int_0^{k/N}\sum_{i=1}^{N} Z_{(i)}\,\xi^i\,d\xi\right]-\frac{k}{N}\,\varepsilon'(s)Z_k(s)+(A\tilde{H})_k-$$

$$(A\tilde{H})_0=\varepsilon(s)\int_0^{k/N}\sum_{i=0}^{N} Y_{(i)}(s)\,\xi^i\,d\xi$$

or

$$\varepsilon(s)\sum_{i=1}^{N} Z'_{(i)}(s)\left(\frac{k}{N}\right)^{i+1}\cdot\frac{1}{i+1}+\varepsilon'(s)\sum_{i=1}^{N} Z_{(i)}(s)\left(\frac{k}{N}\right)^{i+1}\cdot\frac{1}{i+1}-$$

$$\varepsilon'(s)Z_k(s)\frac{k}{N}+(A\tilde{H})_k-(A\tilde{H})_0-\varepsilon(s)\sum_{i=1}^{N} Y_{(i)}(s)\left(\frac{k}{N}\right)^{i+1}\cdot\frac{1}{i+1}=0,$$

$$k=1,\ 2,\ \ldots,\ N.$$

The second equation is transformed similarly. In vectorial form both equations can be written as /160

$$\varepsilon(s)B\bar{Z}'_{(*)}+\varepsilon'(s)B\bar{Z}_{(*)}-\varepsilon(s)B\bar{Y}_{(*)}-\bar{d}=0;$$
$$\varepsilon(s)\bar{m}t'_0+\varepsilon(s)B\bar{t}'_{(*)}+\varepsilon'(s)\bar{m}t_0+\varepsilon'(s)B\bar{t}_{(*)}-\bar{e}=0,$$

where $\bar{B}$, $\bar{d}$, $\bar{e}$ are the matrices

$$B=\|b_{ki}\|;\ b_{ki}=\left(\frac{k}{N}\right)^{i+1}\frac{1}{i+1};\ \bar{m}=\|m_k\|;\ m_k=\frac{k}{N};\ \text{etc.}$$

$$i,k=1,\ 2,\ \ldots,\ N.$$

Using the relationship between coefficients of the approximating polynomials and functions at the approximation lines

$$\bar{Z}_{(*)}=W^{-1}\bar{Z}_*;\ \bar{t}_{(*)}=W^{-1}(\bar{t}_*-\bar{t}_0);\ \bar{Y}_{(*)}=W^{-1}(\bar{Y}_*-\bar{Y}_0):$$
$$W=\|w_{ki}\|;\ w_{ki}=\left(\frac{k}{N}\right)^i,\quad k,i=1,\ \ldots,\ N,$$

we get a system of equations in the form

$$\bar{Z}'_*+\bar{b}=0;\ \bar{a}t'_0+\bar{t}'_*+\bar{c}=0.$$

Here

$$a_i = (WB^{-1}\bar{m})_i - 1; \quad b_i = \left(\frac{\varepsilon'(s)}{\varepsilon(s)}\bar{Z}_* - \bar{Y}_* - \frac{1}{\varepsilon(s)} WB^{-1}\bar{d}\right)_i + Y_0;$$

$$c_i = \left[\frac{\varepsilon'(s)}{\varepsilon(s)}(\bar{t}_* - \bar{t}_0) + \frac{1}{\varepsilon(s)} WB^{-1}(\varepsilon'(s)\bar{m}t_0 - \bar{e})\right]_i, \quad i = 1, \ldots, N.$$

We have thus obtained a system of 2N equations with 2N + 2 unknown functions $(\bar{t}_*, t_0, \bar{Z}_*, \varepsilon)$.

The relationship between the velocity components across the shock wave

$$v_N = w_n \sin(\sigma - \theta); \quad u_N = w_n \cos(\sigma - \theta)$$

yields

$$\alpha v_N' - \beta u_N' + w_n \frac{dw_n}{d\sigma}\theta' = 0; \quad \sigma' = \frac{u_N' - w_n\theta' \sin(\sigma - \theta)}{\alpha},$$

where

$$\alpha = \frac{dw_n}{d\sigma}\cos(\sigma - \theta) - w_n \sin(\sigma - \theta),$$

$$\beta = \frac{dw_n}{d\sigma}\sin(\sigma - \theta) + w_n \cos(\sigma - \theta).$$

Knowing $\sigma(s)$, function $\varepsilon(s)$ is found from the geometrical relationship $\varepsilon'(s) = (1 + \varepsilon/R)\tan(\sigma - \theta)$. Equations for σ' and ε' close the approximating system. /161

Expressing functions t, z, Y, etc. in terms of the sought functions u, v, p and ρ, we get a system which can be written in vectorial form as $\Phi\bar{\varphi}' = \bar{q}$, where

$$\bar{\varphi}=\begin{Vmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \\ u_1 \\ \vdots \\ u_N \end{Vmatrix};\quad \Phi=\left\|\begin{array}{c|ccc|ccc} 0 & n_{11} & & 0 & n_{12} & & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & 0 & & n_{N1} & 0 & & n_{N2} \\ \hline m_{00} & m_{11} & & 0 & m_{12} & & 0 \\ \vdots & & \ddots & & & \ddots & \\ m_{00} & 0 & & m_{N1} & 0 & & m_{N2} \\ \hline 0 & 0 & \dots & 0\alpha & 0 & \dots & 0-\beta \end{array}\right\|;\quad \bar{q}=\begin{Vmatrix} -b_1 \\ \vdots \\ -b_N \\ -c_1 \\ \vdots \\ -c_N \\ -w_n\theta'(s)\dfrac{dw_n}{d\sigma} \end{Vmatrix};$$

$$m_{00}=\frac{r\rho}{c^2}(c^2-v^2)|_0;\quad m_{k1}=\frac{r\rho}{c^2}(c^2-v^2)|_k;\quad m_{k2}=-\frac{\rho uv}{c^2}\Big|_k;$$

$$n_{k1}=\frac{r\rho u}{c^2}(c^2-v^2)|_k;\quad n_{k2}=\frac{r\rho v}{c^2}(c^2-u^2)|_k;\quad c^2=\frac{\rho h_T}{D\rho+2\rho_T}.$$

The total approximating system of equations for numerical integration has the form

$$\left.\begin{array}{l} \bar{\varphi}'=\Phi^{-1}\bar{q};\quad \sigma'=\dfrac{v_N'+w_n\theta'(s)\cos(\sigma-\theta)}{\alpha};\quad \varepsilon'=\left(1+\dfrac{\varepsilon}{R}\right)\operatorname{tg}(\sigma-\theta); \\ w_n'=\dfrac{dw_n}{d\sigma}\sigma';\quad p_N=\dfrac{dp_N}{d\sigma}\sigma';\quad T_N'=\dfrac{dT_N}{d\sigma}\sigma';\quad S_N'=\dfrac{dS_N}{d\sigma}\sigma'; \\ p_0'=-\rho_0 v_0 v_0';\qquad T_0'=\dfrac{1}{h_{T_0}}[h_{p_0}\rho_0-2]v_0 v_0'; \\ \psi_i'=\dfrac{i}{N}\varepsilon'(r\rho v)_i-\left(1+\dfrac{i}{N}\dfrac{\varepsilon}{R}\right)(r\rho u)_i;\quad i=1,2,\dots,N-1; \\ p_i'=-\dfrac{\rho_i}{2}[(w_i^2)'+T_i S_i'];\quad T_i'=-\dfrac{1}{(h_T)_i^2}[(w_i^2)'+ \\ \qquad\qquad h_{T_i}p_i'],\quad i=1,\dots,N-1. \end{array}\right\} \qquad (3.3)$$

The prime everywhere denotes a derivative with respect to s. /162

In scheme I the boundary conditions at the shock wave and the body are satisfied automatically, which is seen from the construction of the approximating system.

For s = 0 we have $v_i(0) = 0$, $\psi_i(0) = 0$, $\sigma(0) = \pi/2$ and $S_i(0) = 0$. The values of $w_n(0)$, $p_i(0)$ and $T_i(0)$ are determined by integrating the equations along the zero streamline from the shock wave to the body, if the values of $u_i(0)$, (i = 1, 2, ...,

N - 1) and $\varepsilon(0)$ are known. It follows from the form of n_k, m_k and m_{00} that for $v_k = c_k$ matrix Φ degenerates at k = 0, 1, ..., N - 1 (this matrix does not degenerate when k = N, due to α). If the solution exists also there, then it may be nonunique. The points at which $v_k = c_k$ are singular points of the system (first-order poles). The set of these points will subsequently be called the singular line of the system. In order that derivatives at points of the singular line be limited, it is necessary to satisfy special conditions for the regularity of solution (for $v_k = c_k$ we whould have $E_k = 0$, where $v_k' = E_k/v_k^2 - c_k^2)$). Otherwise, the derivatives at this line will be infinite, i.e., the accelerations will be infinite and then the motion cannot be extended past the singular line (the singular line proper will serve as a limiting line — the Jacobian $D = \frac{\partial(x, y)}{\partial(w_x, w_y)}$ vanishes along this line).

The nature of these points is clear. The direction s = const at the points is tangent to one of the characteristics of the starting system of differential equations. In fact, at points where v = c, we have $v/w = c/w = 1/M = \sin\alpha$, where α is the Mach number. Depending on the sign of the velocity component u, the direction s = const is tangent, at points of the singular line, to characteristics of the first ($u > 0$) or second ($u < 0$) families.

It is seen from the preceding discussion that in scheme I of the method of integral relations at each integration step one determines the approximate solution of the local Cauchy problem in the vicinity of line s = const for the starting differential operator. The singular line is the locus of points in which the direction of integration becomes perpendicular to the characteristic. The location of singular points in selecting the other sought functions remains the same, whereas changes in the coordinate system (and, finally, the directions along which the derivatives are determined) can also involve changes in the location of the singular line. /163

Thus, the only bounded solution of the starting system of equations is obtained from satisfaction of regularity conditions at points of the singular line for which system (3.3) should be integrated in the region (containing the singular line) up to boundary characteristic $s = s_1(n)$ (here the region of influence is taken into account exactly, Fig. 3.9b) or to the closing ray $s = s_*$ passing through the farthest removed singular point (Fig. 3.9a). The regularity conditions at the N singular points make it possible to uniquely determine the N unknown variables $u_i(0)$, (i = 1, 2, ..., N - 1) $\varepsilon(0)$. The algorithms can be implemented directly on a computer for N = 1, 2, 3. At higher approximations one encounters difficulties related to the need of successive passing of the singular points. In this case one should use an iteration process, in which each iteration is used for passing a single singular point, while functions v_k at the other lines of the subdividing grid are retained in the memory. This iteration scheme was developed at the Computing Center of the USSR Academy of Sciences [29].

Scheme I can be used for calculating flows past blunted bodies with a sharp corner in the contour or with a break in the curvature of the generatrix. The main difficulty arises here in constructing the computational algorithm at the point where the flow swings past the corner. Using for this a polar coordinate system

(Fig. 3.9c), V. F. Ivanov successfully developed a method for calculating this class of bodies and made a large number of calculations. One of his results is shown in Fig. 3.7.

2. Scheme II. We now consider the supersonic flow of a real gas past blunt bodies with allowance for nonequilibrium dissociation [30, 26]. Let the incident flow consist of a μ-component gas mixture containing λ different kinds of atoms. The independent variables in constructing the algorithm using scheme II is the coordinate system s, ξ ($\xi = n/\varepsilon(s)$) (Fig. 3.10a). The complete system contains, in addition to the gasdynamics equations, equations of material balance, the Dalton equation and relaxation equations. The starting system of equations in dimensionless form is written as:

$$\left.\begin{aligned}
&\frac{\partial}{\partial s}\tilde{Z}\varepsilon+\frac{\partial\tilde{M}}{\partial\xi}=\tilde{Y}\varepsilon;\quad \frac{\partial}{\partial s}\tilde{g}\varepsilon+\frac{\partial\tilde{L}}{\partial\xi}=\tilde{X}\varepsilon;\\
&\frac{\partial}{\partial s}\tilde{t}\varepsilon+\frac{\partial\tilde{N}}{\partial\xi}=0;\quad \rho=\frac{p}{T\sum\limits_{i=1}^{\mu}C_i/m_i};\quad w^2+h=1;\\
&h=\sum_{i=1}^{\mu}C_ih_i;\quad \sum_{i=1}^{\mu}G_i^{(k)}C_i=\Delta^{(k)},\quad k=1,2,\ldots,\lambda-1;\\
&\sum_{i=1}^{\mu}C_i=1;\quad \frac{dC_i}{dt}=\frac{\omega_i}{\rho},\quad i=1,2,\ldots,\mu,
\end{aligned}\right\}\qquad(3.4)$$

/164

where

$$\tilde{M}=(A\tilde{H}-\xi\varepsilon_s'Z)r;\quad \tilde{N}=(A\tilde{h}-\xi\varepsilon_s't)r;$$

$$\tilde{L}=(AZ-\xi\varepsilon_s'g)r;\quad \tilde{X}=-\tilde{Z}/R+Ap\sin\theta.$$

Subsequently we represent

$$\tilde{Y}=Y\cos\theta;\quad \tilde{L}=L\cos\theta;\quad \tilde{M}=M\cos\theta;\quad \tilde{N}=N\cos\theta;\quad r=P\cos\theta.$$

The boundary conditions of the problem have been written out in scheme I. In addition, we have at the shock wave $\xi = 1$, $C_i = C_{i\infty}$ and at the boundary chacteristic $s = s_1(\xi)$ the compability equation

$$\frac{v}{w^2}\frac{du}{d\xi}-\frac{u}{w^2}\frac{dv}{d\xi}-\frac{1}{R}\frac{ds}{d\xi}+\frac{\operatorname{ctg}\alpha}{\rho w^2}\frac{dp}{d\xi}+\Gamma_1=0,\qquad(3.5)$$

where

$$\Gamma_1 = \Gamma_0\left[(A\,\mathrm{tg}\,\theta + \xi\varepsilon_s')\frac{ds_1}{d\xi} + \varepsilon\right]\cos\theta;$$

$$\Gamma_0 = \frac{1}{\sin(\bar{\beta}+\alpha)}\left[\frac{1}{r}\sin\bar{\beta}\sin\alpha + \frac{c}{w^2}\sum_{i=1}^{\mu}\left(\frac{h_{C_i}}{Th_T} + \frac{\rho_{C_i}}{\rho}\right)\frac{dC_i}{dt}\frac{R_0}{w_{max}}\right];$$

$$\bar{\beta} = \beta + \theta; \quad \beta = \mathrm{arctg}\,\frac{u}{v}; \quad \alpha = \arcsin\frac{c}{w};$$

$$h_T = \sum_{i=1}^{\mu} C_i(h_i)_T; \quad (h_i)_T = \left(\frac{\partial h_i}{\partial T}\right)_{p,\,C_i}; \quad c^2 = \frac{pTh_T}{\rho Th_T - 2p}.$$

The streamline and characteristic equations have the form

$$\frac{ds_{st.1.}}{d\xi} = \frac{\varepsilon v}{Au - \xi\varepsilon_0' v}; \quad \frac{ds_1}{d\xi} = \frac{\varepsilon}{A\,\mathrm{tg}(\beta \pm \alpha) - \xi\varepsilon_s'}.$$

Writing $\eta = s/s_1(\xi)$, we write along the shock front the approximation of partial differential equations of system (3.4). The set of lines $\eta = k/N$, (k = 0, 1, ..., N) serves as the approximating grid. /165

The functions are represented as

$$f = \sum_{i=0}^{N} f_{(i)}(\xi)\eta^{2i} \qquad \text{for even functions;}$$

$$\cos\theta(s) = \sum_{i=1}^{N} n_{(i)} s^{2i-1};$$

$$f = \sum_{i=1}^{N} f_{(i)}(\xi)\eta^{2i-1} \qquad \text{for odd functions.}$$

The equations of motion and continuity with the approximations made in them are integrated with respect to s from the zero streamline to each of the N remaining lines, for example,

$$\int_0^{(k/N)s_1(\xi)} \frac{\partial}{\partial\xi}\tilde{M}ds + \int_0^{(k/N)s_1(\xi)} \frac{\partial}{\partial s}\tilde{Z}\varepsilon ds - \int_0^{(k/N)s_1(\xi)} \tilde{Y}\varepsilon ds = 0$$

or

$$\sum_{i=0}^{N}\sum_{j=1}^{N} M'_{(i)} n_{(j)} s_1^{2j}\left(\frac{k}{N}\right)^{2(i+j)} \frac{1}{2(i+j)} +$$

$$s'_1 \sum_{i=0}^{N}\sum_{j=1}^{N} M_{(i)} n_{(j)} s_1^{2j-1}\left(\frac{k}{N}\right)^{2(i+j)} \frac{j}{i+j} - s'_1 \tilde{M}_k \frac{k}{N} + (\tilde{Z}\varepsilon)_k -$$

$$\sum_{i=0}^{N}\sum_{j=1}^{N} Y_{(i)} n_{(j)} s_1^{2j}\left(\frac{k}{N}\right)^{2(i+j)} \frac{1}{2(i+j)} = 0, \quad k = 1, 2, \ldots, N.$$

All the other partial differential equations are transformed similarly.

The system of ordinary differential equations in vectorial form, obtained after integration, can be written as

$$G\bar{M}'_{(*)} + H\bar{M}_{(*)} - G\bar{Y}_{(*)} + \bar{e} = 0; \quad R\bar{L}'_{(*)} + S\bar{L}_{(*)} - T\overline{(\tilde{X}\varepsilon)}_{(*)} + \bar{m} = 0;$$

$$G\bar{N}'_{(*)} + H\bar{N}_{(*)} + \bar{p} = 0,$$

where G, H, R, S and T are matrices of the form /166

$$G = \|g_{ki}\|; \quad g_{ki} = \sum_{j=1}^{N} n_{(j)} s_{(j)}^{2j} \frac{k}{N}^{2(i+j)} \frac{1}{2(i+j)},$$

$$i = 0, 1, \ldots, N \text{ etc.}$$

For even functions $\bar{f}_{(*)} = \|f_{(i)}\|$, $i = 0, 1, \ldots. N$. For odd functions $\bar{f}_{(*)} = \|f_{(i)}\|$, $i = 1, 2, \ldots, N$,

$$\bar{e} = \|e_k\|; \quad e_k = \left[(ZP\varepsilon)_k - \frac{k}{N} s'_1 M_k\right]\cos\theta_k;$$

$$\bar{m} = \|m_k\|; \quad m_k = \left[(tP\varepsilon)_k - \frac{k}{N} s'_1 N'_k\right]\cos\theta_k;$$

$$\bar{p} = \|p_k\|; \quad p_k = \left[(gP\varepsilon)_k - \frac{k}{N} s'_1 L_k\right]\cos\theta_k.$$

Here and subsequently the prime denotes differentiation with respect to ξ.

The coefficients in approximations $f_{(i)}$ and the values of functions at boundaries of strips f_i are related as follows for odd and even functions, respectively:

$$\bar{f}_* = W\bar{f}_{(*)}; \quad W = \|w_{j,i}\|; \quad w_{ji} = \left(\frac{i}{N}\right)^{2j-1}, \quad i,j = 1, \ldots, N;$$

$$\bar{f}_{(*)} = M\bar{f}_{(*)}; \quad M = \|m_{j,i}\|; \quad m_{ji} = \left(\frac{i}{N}\right)^{2j}, \quad i,j = 0, 1, \ldots, N.$$

We substitute into the system of equations thus obtained respectively $\bar{f}_{(*)} = W^{-1}\bar{f}_*$; $\bar{f}_{(*)} = M^{-1}\bar{f}_*$, upon which it becomes

$$P\bar{M}_*' + \bar{k} = 0; \quad P\bar{N}_*' + \bar{r} = 0; \quad Q\bar{L}_*' + \bar{s} = 0.$$

Derivatives $\bar{M}_*'$, $\bar{N}_*'$ and $\bar{L}_*'$ are now expressed in terms of derivatives of the sought functions u', v', p' and ρ', a part of them is eliminated using the equation of state, the Bernoulli integral, conditions at the zero streamline and at the characteristic, which yields

$$A_u\bar{u}' + A_v\bar{v}' + A_p\bar{p}' + \bar{a} = 0, \quad B_u\bar{u}' + B_v\bar{v}' + B_p\bar{p}' + \bar{b} = 0;$$
$$C_u\bar{u}' + C_v\bar{v}' + C_p\bar{p}' + \bar{c} = 0,$$

where $\bar{u}' = \|u_i'\|$; $i = 0, 1, \ldots, N$; $\bar{v}' = \|v_i'\|$; $i = 1, \ldots, N$; $\bar{p} = \|p_i\|$; $i = 1, \ldots, N - 1$; A_u, B_u, C_u are $N \times N$-dimensional matrices, Ap, Bp, Cp, are N-dimensional columns. /167

The equations can be written more compactly in the form $\Phi\bar{\varphi}' = \bar{q}$. The system of 3N ordinary differential equations thus obtained can be solved for all the $0 < \xi \leq 1$.

The complete approximating system of equations for numerical integration using scheme II thus has the form

$$\left.\begin{aligned}
&\bar{\varphi}' = \Phi^{-1}\bar{q}; \quad p_0' = \delta_{01}u_0'; \quad p_N' = \delta_{N1}u_N' + \delta_{N2}v_N' + \delta_{N3};\\
&T_i' = -\frac{1}{h_{T_i}}\left[(w_i^2)' + \sum_{j=1}^{\mu} h_{C_{ij}}C_{ij}'\right];\\
&\rho_i = p_i\left(T_i\sum_{j=1}^{\mu}(C_{ij}/m_j)\right)^{-1}, \quad i = 0, 1, \ldots, N;
\end{aligned}\right\} \qquad (3.6)$$

$$
\left.
\begin{gathered}
h_i = \sum_{j=1}^{\mu} h_{ij} C_{ij}, \quad \sum_{j=1}^{\mu} C_{ij} = 1, \quad \sum_{j=1}^{\mu} G_{ij}^{(k)} C_{ij} = \Delta_i^{(k)}, \\
C_{ij}' = (C_i')_{\text{st. 1}} - \left(s_{\text{st. 1}}' - \frac{i}{N} s_1'\right)\left(\frac{\partial C_j}{\partial s}\right); \quad (C_i')_{\text{st. 1}} = \frac{\omega_i}{\rho_i}, \\
i = 0, 1, \ldots, N, \qquad k = 1, 2, \ldots, \lambda - 1.
\end{gathered}
\right\} \qquad (3.6 \text{ cont'd})
$$

The region of integration is here limited from the top by the bounding characteristic. This makes it possible to carry out calculations with the region of integration coinciding identically with the region of influence of the starting system of equations. The scheme under consideration uses the compatibility condition (3.5) along the bounding characteristic. It contains derivatives only along the line $s = s_1(\xi)$, consequently, in the approximating system (3.6) it is used in the exact form in each approximation. This condition completes the approximating system and ensures regularity of the solution of the starting system of differential equations along the upper boundary. The regularity of the solution inside the region of integration follows from the appropriate representation of sought functions along the shock front, regularity of the solution at the boundaries of the region of integration, as well as from the iteration method proper used in constructing the sought numerical solution of the boundary-value problem for the approximating system. /168

For system (3.6) we get a boundary-value problem in the region $0 \leq \xi \leq 1$. The boundary conditions at the axis of symmetry are satisfied automatically, which follows from the construction of the algorithm proper. Specifiying the shock wave by means of $N + 1$ variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$ (for example, by an even-power polynomial), we find the values of all the sought functions at $\xi = 1$ from the applicable conditions at the shock wave. System (3.6) is integrated from $\xi = 1$ to $\xi = 0$, where the values of all the starting variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$. are uniquely determined from the $N + 1$ impermeability conditions at the body ($u_i(0) = 0$, $i = 0, 1, \ldots, N$). The conditions at the body and at the shock wave are here satisfied at discrete points, the number of which increases with the number N of the approximation.

The approximating system in scheme II of the method of integral relations in this problem does not have singular points inside the integration region, since the solution of the local Cauchy problem in the vicinity of line $\xi = \text{const}$ for the starting differential operator is always unique and bounded ($|u| \ll c$).

Variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$ are determined, for example, by Newton's generalized method. Corrections $\delta\varepsilon_j$ to values of ε_j are found from the system of equations $\sum_{j=0}^{N} \frac{\partial u_i}{\partial \varepsilon_j} \delta\varepsilon_j = -u_i(\varepsilon_0, \ldots, \varepsilon_N)$, $i = 0, 1, \ldots, N$, where

$$
\frac{\partial u_i}{\partial \varepsilon_j} \approx \frac{u_i(\varepsilon_0, \ldots, \varepsilon_{j-1}, \varepsilon_j + \Delta, \varepsilon_{j+1}, \ldots, \varepsilon_N) - u_i(\varepsilon_0, \ldots, \varepsilon_j, \ldots, \varepsilon_N)}{\Delta}.
$$

The calculations for flow past bodies with a sonic inflection in the vicinity of the corner point (in region G, Fig. 3.10b) [31] included construction of the Vaglio-Laurin asymptotic solution [32].

The use of the boundary characteristic in scheme II of the method of integral relations makes possible exact closure of the region of integration and of the approximating system. In addition, the boundary conditions here are satisfied exactly in any approximation, which provides for stable computation also with approximations of high orders (the computations were carried out, for example, up to N = 12). /16

3. Scheme III. Here one utilizes double approximation, under which the functions are first represented using scheme II, are integrated, and then approximation is again carried out using scheme I. As a result one obtains an approximating system of nonlinear algebraic equation. The grid used for subdivision of the region of integration is shown in Fig. 3.11.

In scheme II we obtained the system of equations $\bar{\varphi}' = \Phi^{-1}\bar{q}$. We denote $1 - \xi = \zeta$ and introduce the approximations

$$\varphi_i - \varphi_{i0} = \sum_{j=1}^{M} \varphi_{i(j)} \zeta^j,$$

where $\varphi_{i0} = \varphi_i(\zeta = 0)$.

If we designate $\varphi_{ij} = \varphi_i\left(\zeta = \frac{j}{M}\right)$, then $\bar{\varphi}_{i*} - \bar{\varphi}_{i0} = M_0 \bar{\varphi}_{i(*)}$, where $\bar{\varphi}_{i*}$ is a column of rank M with jth element φ_{ij}; $\bar{\varphi}_{i0}$ is a column of the same rank with elements φ_{i0}; $\bar{\varphi}_{i(*)}$ is a column consisting of the coefficients of the approximation, M_0 is a square matrix of rank M and $m_{ij} = (i/N)^j$. From this

$$\bar{\varphi}_{i(*)} = M_0^{-1}(\bar{\varphi}_{i*} - \bar{\varphi}_{i0}).$$

By virtue of the approximations used

$$\varphi_i' = \sum_{j=1}^{M} \varphi_{i(j)} \zeta^{j-1} j \quad \text{or} \quad \bar{\varphi}_{i*}' = \chi \bar{\varphi}_{i(*)},$$

where χ is a square matrix of rank M and $\chi_{ij} = j\left(\frac{i}{M}\right)^{j-1}$. Substituting the expression for $\bar{\varphi}_{i(*)}$, we have

$$\bar{\varphi}_{i*}' = \chi M_0^{-1}(\bar{\varphi}_{i*} - \bar{\varphi}_{i0}).$$

It remains to substitute the approximations thus introduced into the system $\bar{\varphi}' = \Phi^{-1}\bar{q} \equiv \bar{b}$ and write it for all the $\zeta = j/M$, $j = 1, 2, \ldots, M$. We get a system of 3MN nonlinear algebraic equations in the form

$$\chi M_0^{-1} L - B = \chi M_0^{-1} L_0.$$

Here L, L_0 and B are M x 3N matrices, where

$$L = \| l_{ij} \|; \; l_{ij} = \varphi_i\left(\zeta = \frac{j}{M}\right); \; b_{ij} = b_i\left(\zeta = \frac{j}{M}\right); \; l_{0ij} = \varphi_{i0}\left(\zeta = \frac{j}{M}\right). \qquad /170$$

This system is supplemented by the remaining equations of the total approximating system of equations (3.6) of scheme II with the approximations made by setting $\zeta = 1 - \xi$ and written at points of intersection of the computational grid. All the boundary conditions are satisfied at the grid points lying at the boundary of the region.

If through approximation is used (as is done here), then the numerical solution is found by solving a nonlinear algebraic system of equations at all points of the curvilinear grid subdividing the region. The regions of influence are then accounted for exactly. The use of a linear of quadratic approximation yields an ordinary difference scheme on a curvilinear computational grid.

Scheme III can be constructed similarly, first using approximation according to scheme I and then according to scheme II (here it is possible to dispense with the second integration). This version of scheme III was considered in [26, 33]. As was shown by F.D. Popov's calculations (ideal gas, $\varkappa = 1.4$, sphere, $M_\infty = 10$) the exactness of computations using this scheme for M, N = 4-5 (4-5 steps across and along the shock front) is up to 2.5-3 decimal places. The algebraic system was solved by iterations, the time of computation of the complete version on the BESM-2 computer amounting to only a few minutes.

As was mentioned previously, scheme III can be constructed also without the second integration. We shall now present a version of this scheme developed by F.D. Popov [26].

The starting system of partial differential equations can be represented in the form:

$$\frac{\partial}{\partial s} P_i(s, n, U) + \frac{\partial}{\partial n} Q_i(s, n, U) = F_i(s, n, U), \qquad (3.7)$$

where U(s, n) is the system of the sought functions, P_i, Q_i and F_i are known, generally nonlinear functions of s, n, u_i ($i = 1, \ldots, k$). For example, for an equilibrium flow, using in the starting system of equations both projections of the

equation of motion, then k = 3,

$$U \equiv \{u, v, p, \rho, T\},$$

$$P_1 = \bar{t} = r^{\nu}\rho v; \qquad Q_1 = A\bar{I} = Ar^{\nu}\rho u; \qquad F_1 = 0;$$

$$P_2 = \bar{g} = r^{\nu}(p+\rho v^2); \; Q_2 = A\bar{Z} = Ar^{\nu}\rho uv; \qquad F_2 = \bar{X} = -\frac{1}{R}\bar{Z} + \nu AP\sin\theta; \qquad /17$$

$$P_3 = \bar{Z}; \qquad Q_3 = A\bar{H} = Ar^{\nu}(p+\rho u^2); \; F_3 = Y = \frac{1}{R}g + \nu AP\cos\theta.$$

Here $A = 1 + n/R$, R is the radius of curvature of the body, $\nu = 1$ for the axisymmetric and $\nu = 0$ for the planar case.

The system is completed by the Bernoulli integral and the equation of state

$$\left.\begin{aligned} h(p, T) + w^2 &= 1; \\ \rho &= \rho(p, T). \end{aligned}\right\} \tag{3.8}$$

Subsequently, without loss of generality, we shall construct the approximating system according to scheme III (of the type I × II) for Eqs. (3.7) with subscript i omitted.

We subdivide, as in scheme I, the integration region by the lines

$$n = n_k(s) = \xi_k \varepsilon(s); \quad \xi_k = \frac{k}{N}; \quad k = 1, \ldots, N-1$$

and introduce approximations of the sets of functions

$$P(n, s) = \sum_{j=0}^{N} p(s)\xi^j; \quad F(n, s) = \sum_{j=0}^{N} F(s)\xi^j.$$

Integration of the starting system of partial differential equations from the zeroth each of the N remaining lines and the use of the relationship between the coefficients of approximating polynomials and functions at approximation lines yields

$$\Phi\frac{d}{ds}P_* + \Phi\left(\frac{1}{\varepsilon(s)}\frac{d\varepsilon}{ds}P_* - F_*\right) + \frac{1}{\varepsilon(s)}Q_* = 0, \tag{3.9}$$

where $\Phi = AB^{-1}$ is a rectangular (N x N + 1) matrix

$$A = \| a_{kj} \|; \quad a_{kj} = \frac{1}{j+1} \xi_k^{j+1}; \quad k = 1, \ldots, N; \quad j = 0, \ldots, N;$$

B^{-1} is the inverse matrix of

$$B = \| b_{kj} \|; \quad b_{kj} = \xi_k^j; \quad k, j = 0, \ldots, N;$$

$\frac{d}{ds} P_*$, P_*, F_* and Q_*, respectively, are the column matrices

$$\frac{d}{ds} P_* = \left\| \frac{d}{ds} P_k \right\|; \quad k = 0, \ldots, N; \quad P_* = \| P_k \|; \quad k = 0, \ldots, N; \qquad \text{/172}$$

$$F_* = \| F_k \|; \quad k = 0, \ldots, N; \quad Q_* = \left\| Q_k - Q_0 - \xi_k \frac{d\varepsilon}{ds} P_k \right\|; \quad k = 1, \ldots, N;$$

$$f_k = f(s, n_k(s)),$$

and the right-hand side of Eq. (3.9) contains a zero column matrix. Solving system of equations (3.9) for the derivatives of the sought functions (U), we get in the Nth approximation the approximating system according to scheme I.

To obtain the approximating system of scheme III we now draw across the shock front M rays s = const, forming a grid on the region of integration

$$s = s_l = \eta_l s_M; \quad \eta_l = \frac{l}{M}; \quad l = 1, \ldots, M$$

and introduce approximations of functions $P_k(s)$ with respect to s, with allowance for their evenness or oddness relative to the axis of symmetry (for symmetrical flow)

$$P_k(s) = \begin{cases} \sum_{j=0}^{M} {}^{j}P_k \eta^{2j} & \text{for even functions;} \\ \sum_{j=1}^{M} {}^{j}\bar{P}_k \eta^{2j-1} & \text{for odd functions,} \end{cases} \qquad (3.10)$$

where ${}^{j}P_k$ and ${}^{j}\bar{P}_k$ are coefficients of the approximation, defined by values of functions P_k at rays $s = s_l = \text{const}$, $l = 0, \ldots, M$.

Equations (3.10) are substituted in Eqs. (3.9) and the equations thus obtained are written for $l = 0, \ldots, M-1$. Expressing now the coefficients of approximations (3.10) in terms of the values of functions $P_k(s)$ for $s = s_l$ ($l = 0, \ldots, M$), we get a

system of M × N nonlinear algebraic equations in the form

$$\Phi P_{**}\,\bar{\Psi} + \Phi F_{**} + Q_{**} = 0, \tag{3.11}$$

which are the approximate representations, in the given scheme of the partial differential equations of the starting system (3.7). Here

$$P_{**} = \frac{1}{s_M} \| P_{kl} \|; \qquad k = 0, \ldots, N;$$

$l = 0, \ldots, M$ for even and $l = 1, \ldots, M-1$ for odd functions

$$F_{**} = \left\| \frac{1}{\varepsilon_l} \left(\frac{d\varepsilon}{ds}\right)_l P_{kl} - F_{kl} \right\|; \quad k = 0, \ldots, N; \quad l = 0, \ldots, M-1; \qquad /17$$

$$Q_{**} = \left\| \frac{1}{\varepsilon_l} \left[Q_{kl} - Q_{0l} - \xi_k \left(\frac{d\varepsilon}{ds}\right)_l P_{kl} \right] \right\|; \; k = 1, \ldots, N; \; l = 0, \ldots, M-1;$$

$\bar{\Psi}$ is the transpose of matrix Ψ, which is given as

$$\Psi = C \cdot D^{-1};$$

D^{-1} is the inverse of matrix D

$$D = \| d_{lj} \|;$$

$$d_{lj} = \begin{cases} \eta_l^{2j}; & l, j = 0, \ldots, M \text{ for even functions;} \\ \eta_l^{2j-1}; & l, j = 1, \ldots, M \text{ for odd functions;} \end{cases}$$

$$C = \| C_{lj} \|;$$

$$C_{lj} = \begin{cases} 2j\eta_l^{2j-1}; \; l = 0, \ldots, M-1; \; j = 0, \ldots, M & \text{for even functions;} \\ (2j-1)\eta_l^{2j-2}; \; l = 0, \ldots, M-1; \; j = 1, \ldots, M & \text{for odd functions.} \end{cases}$$

The right-hand side of Eq. (3.11) contains an (N × M)- zero matrix.

Approximating system of equations (3.11) of scheme III together with the finite difference equations (3.8) determines the values of the sought functions in the

internal nodes of the region of integration. The boundary conditions on the wave and on the body are satisfied automatically, since they are incorporated in constructing scheme (3.9) (see subsection 2 of this section III). The conditions of regularity of solution in singular points of system (3.9) and the conditions at the axis of symmetry complete the approximating system of equations. Since in general rays $s = s_l$ do not pass through singular points, then it is better to replace conditions of the type: $E_k = 0$ when $v_k = c_k$ by regularity conditions, written as expansions in s along each of the lines

$$\begin{gathered} \xi_k \quad (k = 0, \ldots, N-1); \\ \sum_{j=0}^{M} {}^{j}E_k \eta_m^{2j} = 0; \quad k, m = 0, \ldots, N-1, \end{gathered} \tag{3.12}$$

where ${}^{j}E_k$ are coefficients of the approximation, defined by the values of $E_k(s)$ /174 defining the location of singular points at lines ξ_k. Quantities η_m, which are not known a priori, are determined from N equations for

$$\begin{gathered} N_k(s) = v_k^2 - c_k^2; \\ \sum_{j=0}^{M} {}^{j}N_k \eta_m^{2j} = 0, \quad k, m = 0, \ldots, N-1. \end{gathered} \tag{3.13}$$

similar to Eqs. (3.12).

If system of equations (3.11) is written for all the rays $s = s_l$ ($l = 0, \ldots, M$), then the approximating system of scheme III will become intedeterminate and Eqs. (3.12) and (3.13) could then be used for checking the calculations.

The approximating equations of scheme III of the method of integral relations were used successfully, for example, in calculations of supersonic gas flow at low M_∞ (the calculations were carried out up to $M_\infty = 1.05$). Some results of these calculations are presented later on. Schemes of this kind were found sufficiently effective in calculations of viscous gas flows, as well as of flows with radiation.

In essance, schemes III of the method of integral relations are finite-difference schemes. Here, however, one uses a curvilinear computational grid (isolating the shock front), the directions of approximations and expressions for the "complexes" of functions being represented being selected from the requirement that they vary as smoothly as possible. The finite difference schemes are thus constructed here with special computational grids, when the starting system of partial differential equations is written in divergence form.

4. In the general case of nonlinear partial differential equations it is quite difficult to obtain analytic estimates of accuracy and convergence of the method of

integral relations. Hence all kinds of numerical estimates should be regarded here as the main criterion. For this purpose each of the schemes of the method of integral relations was used for calculations in various approximations, and results of computations obtained by these different schemes were compared.

Calculations using scheme I were carried out for N = 1, 2, 3. The experience of calculations shows that sufficient accuracy (2.5-3 decimal places) using this scheme is obtained in the majority of cases for axisymmetrical bodies with N = 2 and for flat bodies with N = 2, 3.

The algorithm of scheme II of the method of integral relations was considered in detail. Calculations for various external flow regimes (perfect gas; equilibrium, nonequilibrium flows; various body shapes) were carried out up to approximations 7-12 [26, 28]. Some of the computed results of A. Bulekbayev using this scheme for the approximations N = 2, 3, ..., 6 (sphere, $M_\infty = 10$, perfect gas, $\varkappa = 1.4$) are presented in Tables 1 and 2. The computed results (detachment of [bow] shock wave, velocity components, density and temperature) are presented for the region between the shock wave and the body $0 \leq \xi \leq 1$ along axis of symmetry $s = 0$ and the bounding characteristic of the second family $s = s_1(\xi)$. It was established in calculations for flows past smooth bodies using scheme II of the method of integral relations for a perfect gas, as well as with allowance for equilibrium or nonequilibrium processes, that the accuracy of the method increases inside the region of intergration by approximately one decimal place in each new approximation (from 3.5-3 decimal places with N = 2-3 to 4-6 decimal places with N = 5-6).

TABLE 1. DETACHMENT OF [BOW] SHOCK WAVE (APPROXIMATE COMPUTATIONS USING SCHEME II), $M_\infty = 10$, SPHERE, PERFECT GAS, $\varkappa = 1.4$.

$\varepsilon(s)$	N				
	2	3	4	5	6
$\varepsilon(0)$	0.135623	0.135717	0.135718	0.135718	0.135718
$\varepsilon(s_1)$	0.173651	0.174606	0.174607	0.174607	0.174607

As another criterion of a method's accuracy one can compare the results obtained using different schemes. Table 3 presents data obtained with schemes I and II (N = 2) in calculating flows of an ideal gas past a sphere ($M_\infty = 10$, $\varkappa = 1.4$). Table 4 compares the calculated results of F.D. Popov using schemes I (N = 2) and III (N = 3, M = 2-5) for the same case. As can be seen from these comparisons, satisfactory agreement exists between results obtained with both schemes, although each of them uses approximate representation of functions in diametrically opposite directions. This points to reliability of the results and to the fact that the starting differential operator is approximated with sufficient accuracy.

TABLE 2. DISTRIBUTION OF GAS VARIABLES ALONG THE AXIS OF SYMMETRY S = 0 (APPROXIMATE COMPUTATIONS USING SCHEME II) $M_\infty = 10$, SPHERE, PERFECT GAS, $\varkappa = 1.4$.

ξ	N	u	v	p	ρ	T
1.00	2	-0.170781	0	0.792517	5.714286	0.138690
1.00	3	-0.170782	0	0.792517	5.714285	0.138690
1.00	4	-0.170782	0	0.792517	5.714285	0.138691
1.00	5	-0.170783	0	0.792517	5.714285	0.138691
1,00	6	-0.170783	0	0.792517	5.714285	0.138691
0.75	2	-0.124069	0	0.832573	5.917912	0.140658
0.75	3	-0.124210	0	0.832466	5.918584	0.140653
0.75	4	-0.124209	0	0.832470	5.918601	0.140653
0.75	5	-0.124209	0	0.832470	5.918602	0.140653
0.75	6	-0.124209	0	0.832470	5.918602	0.140653
0.50	2	-0.080801	0	0.859074	6.053099	0.141923
0,50	3	-0.081050	0	0.858978	6.052615	0.141919
0.50	4	-0.081047	0	0.858983	6.052614	0.141919
0.50	5	-0.081046	0	0.858984	6.052613	0.141919
0.50	6	-0.081046	0	0.858984	6.052613	0.141919
0.25	2	-0.039857	0	0.874149	6,128028	0.142630
0.25	3	-0.039962	0	0.874119	6,128051	0.142629
0.25	4	-0.039964	0	0.874123	6,128053	0.142629
0.25	5	-0.039961	0	0.874124	6,128053	0.142629
0.25	6	-0.039961	0	0.874124	6,128053	0.142629
0.00	2	-0.000017	0	0.879103	6.153568	0.142861
0.00	3	-0.000002	0	0.879102	6.153569	0.142861
0.00	4	-0.000024	0	0.879103	6.153569	0.142861
0.00	5	-0.000029	0	0.879103	6.153569	0.142861
0.00	6	-0.000029	0	0.879103	6.153569	0.142861

TABLE 3. COMPARISON OF COMPUTATIONAL RESULTS USING SCHEMES I AND II (N = 2), $M_\infty = 10$, SPHERE, PERFECT GAS, $\varkappa = 1.4$.

	Schemes	Coordinates of points								
		ξ = 1,0 s = 0	ξ = 0,5 s = 0	ξ = 0 s = 0	ξ = 1,0 s = 0,3250	ξ = 0,5 s = 0,3450	ξ = 0 s = 0,3575	ξ = 1,0 s = 0,8500	ξ = 0,5 s = 0,8900	ξ = 0 s = 0,7151
ε	I	0,1360	–	–	0,1444	–	–	0,1744	–	–
	II	0,1356	–	–	0,1438	–	–	0,1736	–	–
u	I	-0,1710	-0,0782	0	-0,1515	-0,0728	0	-0,0899	-0,0417	0
	II	-0,1708	-0,0809	0	-0,1522	-0,0728	0	-0,0906	-0,0116	0
v	I	0	0	0	0,2742	0,2431	0,2078	0,5084	0,4523	0,4091
	II	0	0	0	0,2757	0,2431	0,2088	0,5088	0,4532	0,4083
p	I	0,7930	0,8000	0,8793	0,7338	0,7614	0,7526	0,5892	0,5332	0,4628
	II	0,7925	0,8591	0,8791	0,7325	0,7610	0,7517	0,5877	0,5343	0,4648
ρ	I	0,1387	0,1420	0,1129	0,1288	0,1337	0,1366	0,1048	0,1134	0,1169
	II	0,1387	0,1413	0,1129	0,1287	0,1336	0,1366	0,1046	0,1133	0,1160

TABLE 4. COMPARISON OF COMPUTATIONAL RESULTS USING SCHEME III (UTILIZING APPROXIMATIONS) AND SCHEME I (N = 2) $M_\infty = 10$, PERFECT GAS, $\varkappa = 1.4$, SPHERE.

	M	s	ξ = 1				ξ = 0,5			ξ = 0		
			ε	p	u	v	p	u	v	p	u	v
Scheme III	2	0	0,1507	0,7925	−0,1708	0	0,8579	−0,06329	0	0,8790	0	0
	3	0	0,1351	0,7925	−0,1708	0	0,8588	−0,06148	0	0,8790	0	0
	4	0	0,1351	0,7925	−0,1708	0	0,8588	−0,06151	0	0,8790	0	0
	5	0	0,1354	0,7925	−0,1708	0	0,8587	−0,06170	0	0,8790	0	0
Scheme I		0	0,1360	0,7930	−0,1798	0	0,8580	−0,06190	0	0,8790	0	0
Scheme III	2	0,250	0,1554	0,7515	−0,1621	0,2252	0,8002	−0,07859	0,1602	0,8428	0	0,1093
	3	0,250	0,1395	0,7556	−0,1602	0,2157	0,8048	−0,07690	0,1791	0,8152	0	0,1460
	4	0,250	0,1398	0,7556	−0,1601	0,2153	0,8050	−0,07691	0,1790	0,8153	0	0,1458
	5	0,250	0,1397	0,7556	−0,1602	0,2158	0,8047	−0,07685	0,1787	0,8149	0	0,1463
Scheme I		0,250	0,1410	0,7570	−0,1600	0,2130	0,8090	−0,07890	0,1780	0,8150	0	0,1460
Scheme III	2	0,500	0,1648	0,6508	−0,1298	0,4218	0,6532	−0,06378	0,3319	0,6813	0	0,2650
	3	0,500	0,1551	0,6596	−0,1256	0,4105	0,6657	−0,06225	0,3463	0,6357	0	0,2974
	4	0,500	0,1553	0,6591	−0,1258	0,4110	0,6648	−0,06223	0,3464	0,6361	0	0,2942
	5	0,500	0,1553	0,6591	−0,1258	0,4110	0,6652	−0,06234	0,3462	0,6361	0	0,2950
Scheme I		0,500	0,1570	0,6620	−0,1250	0,4080	0,6670	−0,06240	0,3430	0,6170	0	0,2900
Scheme III	2	0,625	0,1768	0,5929	−0,1007	0,5019	0,5697	−0,05206	0,4082	0,5311	0	0,3682
	3	0,625	0,1684	0,5960	−0,09914	0,4936	0,5750	−0,04908	0,4238	0,5139	0	0,3560
	4	0,625	0,1685	0,5963	−0,09900	0,4933	0,5751	−0,04920	0,4235	0,5385	0	0,3515
	5	0,625	0,1685	0,5962	−0,09905	0,4931	0,5750	−0,04919	0,4235	0,5385	0	0,3515
Scheme I		0,625	0,1710	0,6020	−0,09650	0,4930	0,5800	−0,04990	0,4170	0,5420	0	0,3750

The accuracy of the method can also be estimated by checking, in the process of numerical solution, of known exact relations, which are not used in constructing the approximating scheme (such as the integrals of the starting system, equations of constant flowrate, constant entropy along streamlines, etc.). Calculations show that for scheme I in the second approximation the condition of constant flowrate is satisfied for any streamtubes with an error up to 0.1%, while in scheme II, where the entropy is not used in the starting system of equations, $S = S_*(\psi)$ is satisfied to within the second approximation with an error up to 1%.

Figure 3.12 also illustrates the accuracy of schemes of the method of integral relations. Figures 3.12a and b compare the computed results for a sphere using scheme I ($M_\infty = 4$, $\varkappa = 1.4$) and of calculations by the finite difference scheme of [34]. As is seen, when the mesh is made "finer" the results obtained by the finite difference method approach those obtained by the method of integral relations. Figure 3.12 shows the variation in the location of the shock wave $\varepsilon(s)$ and the pressure distribution along a sphere obtained from computations with different approximations (N = 1, 2, 3) by means of schemes I and II. /178

These estimates show that the method converges rapidly and also that the results using both schemes become virtually identical as early as in the second or third approximation.

The above algorithms were developed for different cases of flow by the authors of [26].

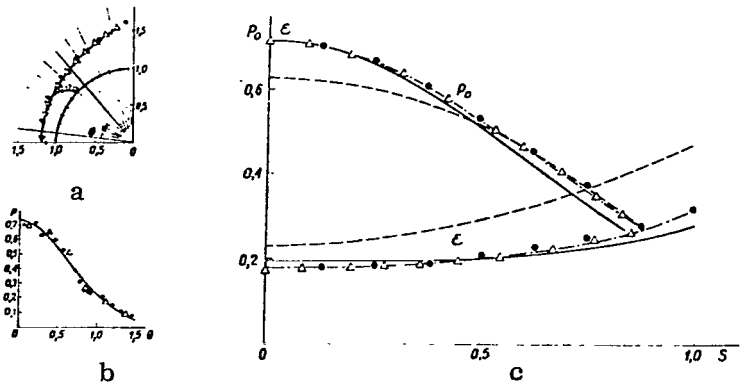


Figure 3.12.

a) and b) Sphere, $M_\infty = 4$, $\varkappa = 1.4$; —— - Method of Integral Relations (N = 2); ΔΔΔ - Computation "A", Method of Finite Differences; ... - Computation "B" Method of Finite Differences; c) Scheme I (... - N = 2), Scheme II (--- - N = 1, $s_{10} = 1.25$: —— - N = 2, $s_{10} = 1.25$: -.- - N = 2, s = 1: ΔΔΔ - N = 3, s = 1); $\varkappa = 1.4$.

5. The above numerical schemes of the method of integral relations have come into extensive use in computations of supersonic flow past bodies in the USSR and abroad. The method has been sufficiently effective in the study of the flow pattern for bodies of various shapes. The use of standard programs of algorithms described above makes possible serial calculations of flows past bodies with detached [bow] /179 shock wave with sufficient accuracy on medium-capacity computers.

We shall now discuss individual studies carried out lately and shall present some of their results. More details can be found in the cited references.

Scheme I [23] was used for developing a method for calculating equilibrium gas flows. The calculations were made for various gas mixtures. In particular, bodies with a sharply varying curvature were considered. Figure 3.13a depicts shapes of shock waves and sonic lines, while Fig. 3.13b displays the distribution of pressure $\bar{p}_0$ and temperature $\bar{T}_0$ (referred to the corresponding values in the stagnation point) over the surface of a sphere ($\delta = b/a = 1$) and ellipsoids of revolution with different blunting radius. It is seen that with an increase in the blunting radius (in δ) the flow variables in the subsonic domain vary increasingly slowly; however, here the manner in which quantity $\varepsilon(s)$ - detachment of the shock wave from the body, behaves changes.

Calculations for equilibrium gas flows using schemes I and II with allowance for /180 ionization were carried out by Yu. P. Lun'kin, F.D. Popov and T. Ya. Timofeyeva [26]. Figure 3.14 displays curves of ε_0, density and mass concentrations of O_2, O and O^+ (for oxygen) and molecular concentration of O for air in the stagnation

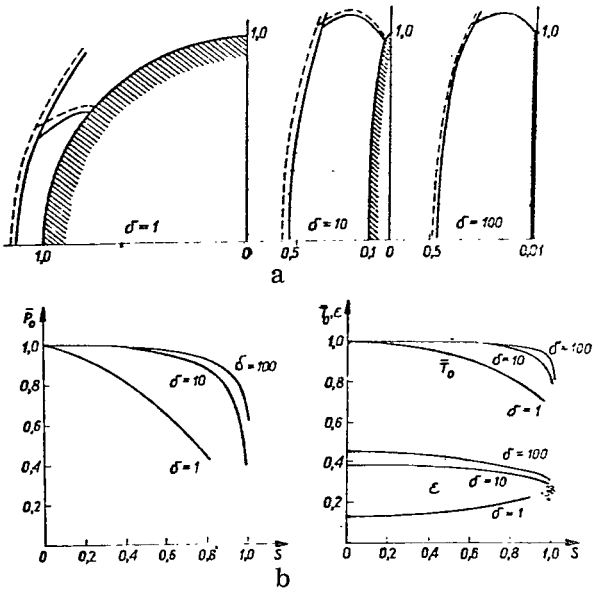


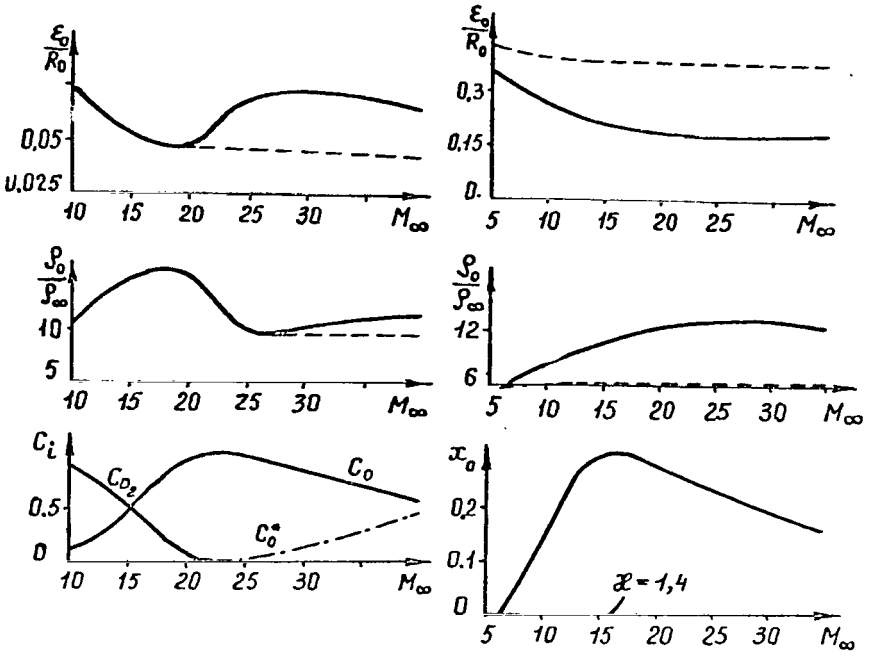
Figure 3.13.

$M_\infty = 6$, $\nu = 1$, $p_\infty = 1$ atm, $T_\infty = 300^\circ$K,
a) --- $\varkappa = 1.4$, —— - Air at Equilibrium;
b) Air at Equilibrium.

point of the body as a function of M_∞. Computational results without ionization and for a perfect gas with $\varkappa = 1.4$ are presented for comparison. The nonmonotonic behavior of $\varepsilon_0(M_\infty)$, which is "spoon-shaped", is determined by the nonmonotonic nature of density variation, which is due to different contributions made by the vibrational energy, dissociation and ionization to the enthalpy of the gas (which is particularly clearly seen for oxygen). Comparison of experimental data on the distance between a detached [bow] shock wave and the body with computed results can thus serve as a good criterion in the study of various physio-chemical processes occurring across the shock front. /181

An item of great interest is the computation of flow past blunt bodies with a discontinuity in the curvature, or sharp corner in the generatrix. For small "rounding" radii $\bar{r}$ (referred to the middle section of the body) or a sharp corner in the contour, the values of some derivatives will have a discontinuity at ray $s = \hat{s}$, where the curves forming the contour or generatrix join, and this involves difficulties in computations, if smoothing is not carried out beforehand.

M. M. Golomazov [26] used scheme I for a series of calculations (for an ideal as well as equilibrium-dissociated gases) for cylindrical bodies with roundings $1/\bar{r}$ ranging from 1 to 64. Figure 3.15 shows flow patterns and the pressure and temperature distributions along the body. One's attention is attracted by the peculiar shape of the sonic line for $1/\bar{r} = 64$, which is due to a sharp swing in the flow in the vicinity of the sonic point. /182



a b

Figure 3.14.

a) Equilibrium Flow of Oxygen, $\delta = 1$, $\nu = 1$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$, —— - Calculated with Ionization, --- - Calculated without Ionization, b) Equilibrium Flow of Air, $\delta = 10$, $\nu = 1$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$, —— - Calculated with Ionization; --- - $\varkappa = 1.4$.

V.F. Ivanov [26, 35] utilized scheme I and the method of characteristics for developing a general method of integrated computation of mixed as well as of supersonic flow across the shock front. He was the first to construct and implement a computational algorithm for flow past bodies with a sonic inflection in the generatrix. The swing in the flow in the vicinity of the inflection, where the Prandtl-Meyer solution holds, is computed by him using a differential equation, which is the compatibility condition along the characteristic of the second family. Figures 3.16 and 3.17 illustrate Ivanov's calculations. It is interesting to note the appearance in the supersonic zone of a "suspended" shock (which was verified experimentally), the strength of which decreases rapidly as the distance from the body's nose increases. In a number of cases the calculations were continued for tens of rounding diameters, which was possible only when the initial data on the bounding characteristic were obtained with high accuracy.

Reference [30] presents a description of scheme II of the method of integral relations, successfully used by V.K. Dushin for calculating nonequilibrium gas flows. Considered are different conditions in the incident flow, multicomponent gas

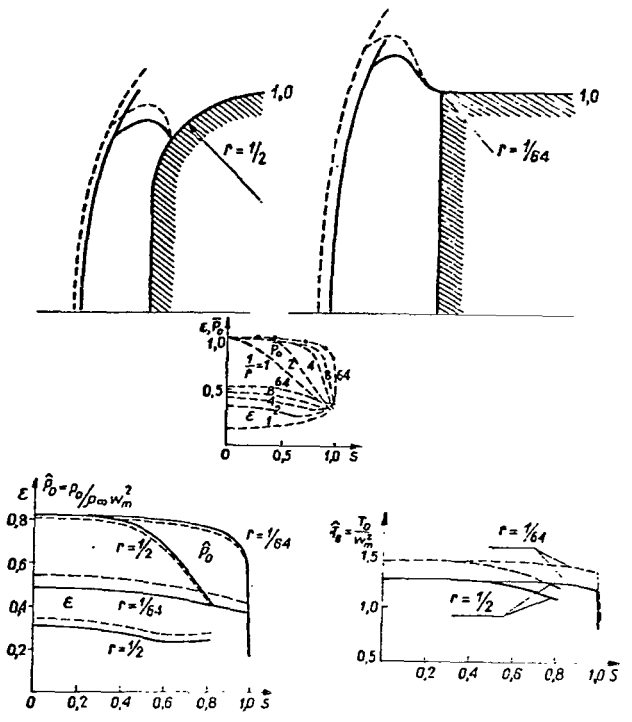


Figure 3.15.

$M_\infty = 5.8$; $\nu = 1$, —— - Air at Equilibrium; $p_\infty = 1$ atm; $T_\infty = 300^\circ K$; --- - $\varkappa = 1.4$; . . . - Experimental Data of Fraasa.*

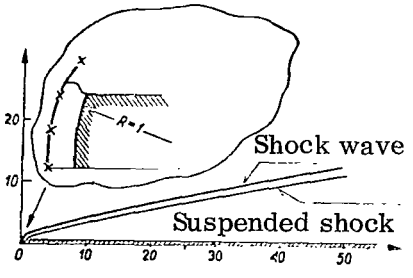


Figure 3.16. Spherical Rounding, $\nu = 1$; $M_\infty = 5.8$; $\hat{s} = 33^\circ$, $\omega = 0^\circ$; $\varkappa = 1.4$; —— - Computed; x x x - Experimental Data of Fraasa.

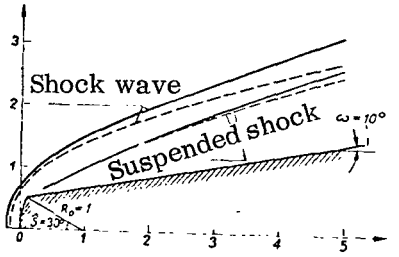


Figure 3.17. Spherical Rounding, $\nu = 1$; $\hat{s} = 30^\circ$; $\omega = 10^\circ$; $\varkappa = 1.4$; —— - $M_\infty = 4$; --- - $M_\infty = 6$.

*Fraasa, D., An Experimental Investigation of Hypersonic Flow Over Blunt-Nosed Bodies at a Mach Number of 5.8. GALCIT Report No. 2, 1957.

mixtures, as well as the effect of individual reactions and of the dimensions of the body on the results. V. K. Dushin carried out calculations in the supersonic region also by the method of characteristics; here the numerical scheme developed by him /183
[19] makes it possible to make computations for equilibrium as well as nonequilibrium flows using the same programs. Figures 3.18 and 3.19 (for a sphere) and 3.20 and 3.21 (for a cylindrical body with spherical rounding with a sharp corner in the contour at $\hat{s} = 0.77$) show the pressure, temperature and concentration distributions for atomic oxygen along the body's surface.

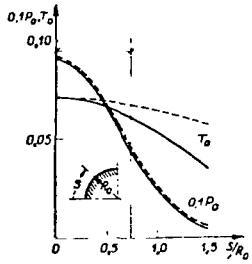


Figure 3.18.

$M_\infty = 10$, $p_\infty = 0.001$ atm; $T_\infty = 288^\circ K$; $R_0 = 1$ m; —— - Nonequilibrium Oxygen; --- - Equilibrium Oxygen.

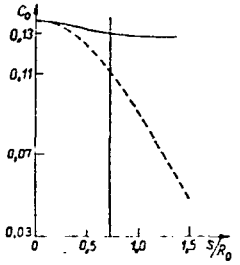


Figure 3.19.

$M_\infty = 10$, $p_\infty = 0.001$ atm; $T_\infty = 288^\circ K$; $R_0 = 1$ m; —— - Nonequilibrium Oxygen; --- - Equilibrium Oxygen.

It follows from this that departure from equilibrium of chemical transformation brought about by a sharp reduction in the particle velocities when crossing the shock wave has almost no effect, at the surface of a smooth body, on the pressure (density, /184
flow velocity), but a perceptible departure is observed of the temperature and concentration of components from their equilibrium values (see Figs. 3.18 and 3.19). At the same time the departure from equilibrium produced by the sharp velocity rise near the sharp corner in the body results in a pronounced departure from equilibrium values of all the flow variables (see Figs. 3.20 and 3.21).

V. K. Dushin and Yu. P. Lun'kin [36] applied scheme II to the study of the effect of an ensemble of various reactions in nonequilibrium air on the distributions of flow variables across the shock front and at the body's surface. It follows from Fig. 3.22a that the distributions of dimensionless p, T and u along the axis of symmetry (and consequently also the location of the shock wave) in the flow of nonequilibrium air past a sphere are almost independent of the number of reactions for which allowance is made. At the same time the component concentrations are highly sensitive to this (Fig. 3.22b). Figure 3.22c shows the pressure, temperature and velocity distri- /185
butions along the sphere surface. Figure 3.22d, e and f shows the variations in all the concentrations along axis of symmetry $s = 0$ of an intermediate line $s = s_1(\xi)/2$

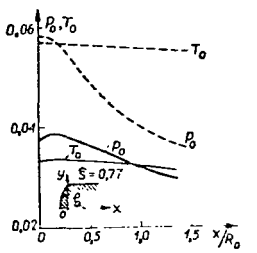


Figure 3.20.

$M_\infty = 10$, $p_\infty = 0.001$ atm; $T_\infty = 288^\circ K$; $R_0 = 1$ m; —— - Nonequilibrium Oxygen; --- - Equilibrium Oxygen.

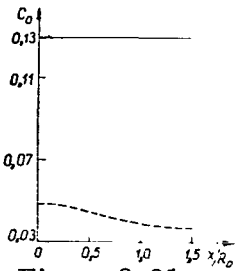


Figure 3.21.

$M_\infty = 10$, $p_\infty = 0.0001$ atm; $T_\infty = 288^\circ K$; $R_0 = 1$ m; —— - Nonequilibrium Oxygen; --- - Equilibrium Oxygen.

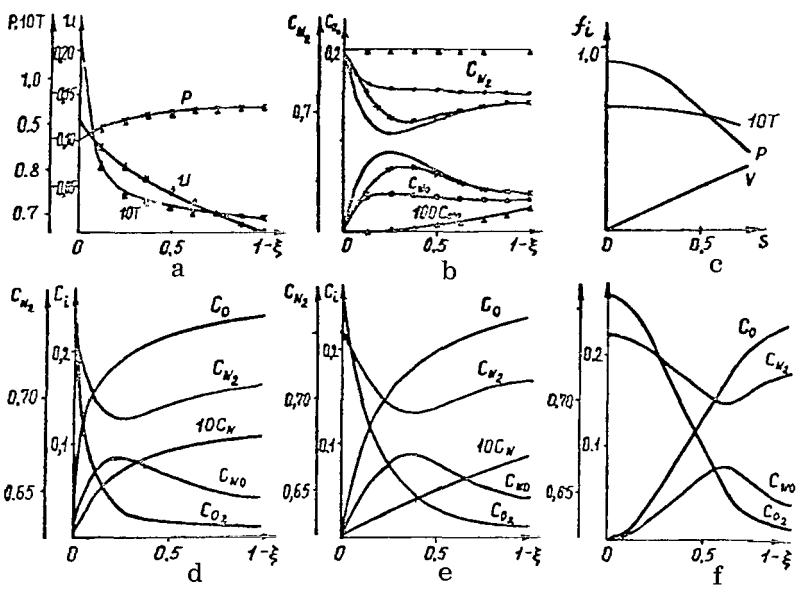


Figure 3.22. Nonequilibrium Air, $M_\infty = 15$, $p_\infty = 0.01165$ atm, $T_\infty = 231^\circ$, 24 K, $(C_{O_2})_\infty = 0.2646$, $(C_{N_2})_\infty = 0.7354$, $(C_{Ar})_\infty = 0$, $R_0 = 1$ cm, $\delta = 1$ (Sphere); —— - Six Reactions; —×—×— without Reaction VI; —o—o— without Reaction V; —▲—▲— - without Reactions IV-VI; a) $s = 0$; b) $s = 0$; d) $s = 0$; e) $s = s_1/2$; f) $s = s_1$.

and the bounding characteristic $s = s_1(\xi)$. An allowance was made for all the six reactions:

$O_2 + M \rightleftarrows 2O + M$; I $\quad N + O_2 \rightleftarrows NO + O$; IV

$N_2 + M \rightleftarrows 2N + M$; II $\quad N_2 + O \rightleftarrows NO + N$; V

$NO + M \rightleftarrows N + O + M$; III $\quad N_2 + O_2 \rightleftarrows 2NO$. VI

The N_2 concentration is at minimum and that of NO is at maximum; these concentrations move from the shock wave to the body with increase in s.

Yu. P. Lun'kov and F. D. Popov were able to develop an algorithm of scheme I for calculating flows of nonequilibrium gas mixtures [37], which involved difficulties due to the unstable approximation of the second equation of motion. They have also investigated the simultaneous occurrence of nonequilibrium excitation of vibrations and of dissociation in a diatomic gas and considered the effect of these phenomena on supersonic flow past blunt-nosed bodies [38].

Here are some of the results obtained by them. Figure 3.23 depicts the flow patterns for nonequilibrium oxygen (equilibrium vibrations) for a sphere ($\delta = 1$) and ellipsoids of revolution with $\delta = 10$ and 100. The dashes denote streamlines, and the dash-dot lines (for the sphere) denote the line of "local equilibrium", with dissociation predominating downstream and recombination predominating upstream of it. It is seen that the region far from the stagnation point, where recombination has a pronounced effect becomes quite perceptible. This is due to sharp flow expansion in this region. Figure 3.23 also depicts the pressure and temperature distribution along the surface of a sphere with radius $R_0 = 4$ cm (curve 1) and $R_0 = 10$ cm (curve 2). The absence of dissociation equilibrium has a marked effect on the temperature distributions (and on the degree of dissociation) and little effect on the pressure distribution; here these distributions tend to equilibrium with an increase in the sphere radius (curve 3).

Figure 3.24 shows the location and shapes of shock waves and sonic lines for a sphere in the case of fully equilibrium flow of oxygen (curve 1), flow with nonequilibrium dissociation but equilibrium vibrations (curve 2* for $R_0 = 10$ cm and 2 for $R_0 = 1$ cm), flow with nonequilibrium dissociation and nonequilibrium vibrations (curve 3) and "frozen" flow for $\varkappa = 1.4$ (curve 4). One's attention is attracted by the characteristic shape of the sonic line in the case of vibrational-dissociation relaxation. Figure 3.25 shows the temperature change from the wave to the body along the axis of symmetry for the same cases (the change is greatest when vibrational-dissociation relaxation is taken into account). Figure 3.26 shows the distribution of t, the average energy of vibrations across the shock front at rays s = const, while Fig. 3.27 depicts the distributions of translational T and vibrational T_v temperatures along the axis of symmetry. It is seen that vibrational degrees of freedom become highly excited immediately behind the wave (which also involves a rise in T_v), then dissociation starts developing and this results in reduction in the translational and vibrational temperatures. Complete equilibrium ensues only in the

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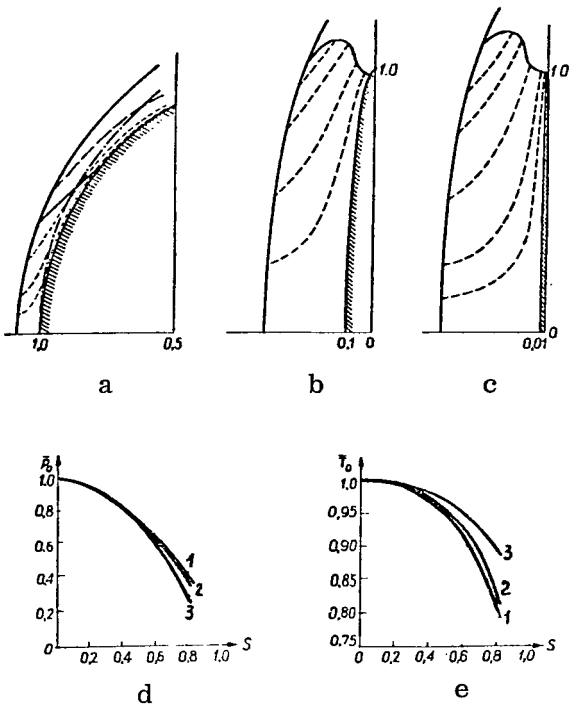


Figure 3.23. Nonequilibrium dissociation, Equilibrium Vibrations, $M_\infty = 10$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$.

a) $R_0 = 10$ cm, $\delta = 1$; b) b = 1 cm, $\delta = 10$; c) b = 1 cm, $\delta = 100$; d) and e) $\delta = 1$.

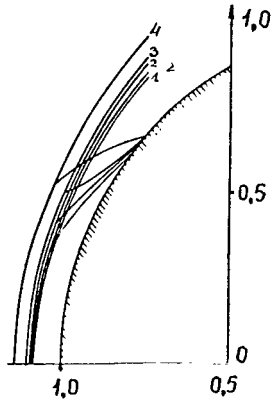


Figure 3.24.

$M_\infty = 10$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$.

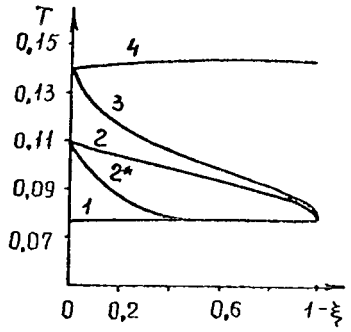


Figure 3.25.

$M_\infty = 10$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$, s = 0, $R_0 = 1$ cm.

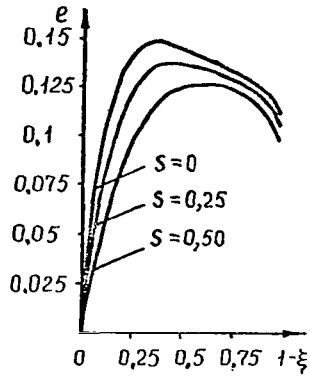


Figure 3.26.

$M_\infty = 10$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$.

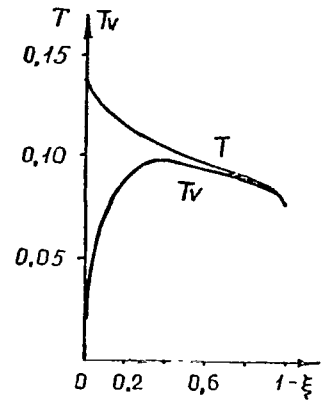


Figure 3.27.

$M_\infty = 10$, $p_\infty = 0.01$ atm, $T_\infty = 290^\circ K$, $s = 0$.

stagnation point, with T_v always remaining smaller than T. For the case when the effect of nonequilibrium excitation of vibrational degrees of freedom on the rate of dissociation was not considered (i.e., a higher [than actual] velocity was considered), the distributions of T and T_v had the same nature as Fig. 3.27, except that a region existed where $T_v > T$. Quantity e, as well as T_v, has a maximum which decreases and moves toward the body as one moves away from the axis of flow. This is due to the fact that the gas temperature decreases upstream and this results, on one hand in increasing the relaxation time for excitation of vibrational degrees of freedom, and on the other hand in reducing the magnitude proper of the vibrational energy. /188

V.N. Fomin [39] developed a scheme for computing flows with radiation. Figure 3.28 illustrates his results for a flow with equilibrium radiation without absorption, i.e., A = 0 (optically thin medium). He considered the flow of radiating air past a sphere. The temperature, pressure and velocity distributions along axis of symmetry s = 0 are given (these data have subscript 0), along the intermediate line $s = s_1(\xi)/2$ (subscript 2) and boundary characteristic $s = s_1(\xi)$ (subscript 1). One observes a marked change in temperature (and, consequently, also in the stagnation enthalpy) and a relatively small change in pressure as compared with the case without radiation (E = 0). The radiative energy E was calculated by approximating Kivel's data [26]. /189

A.I. Tolstykh [40] examined the case of supersonic flows past blunt-nosed bodies at low Re (for example, for flight in a rarefied atmosphere), where the usual subdivision of the entire flow into external inviscid flow and a boundary layer becomes invalid. The entire flow is regarded as viscous, obeying the Navier-Stokes law.

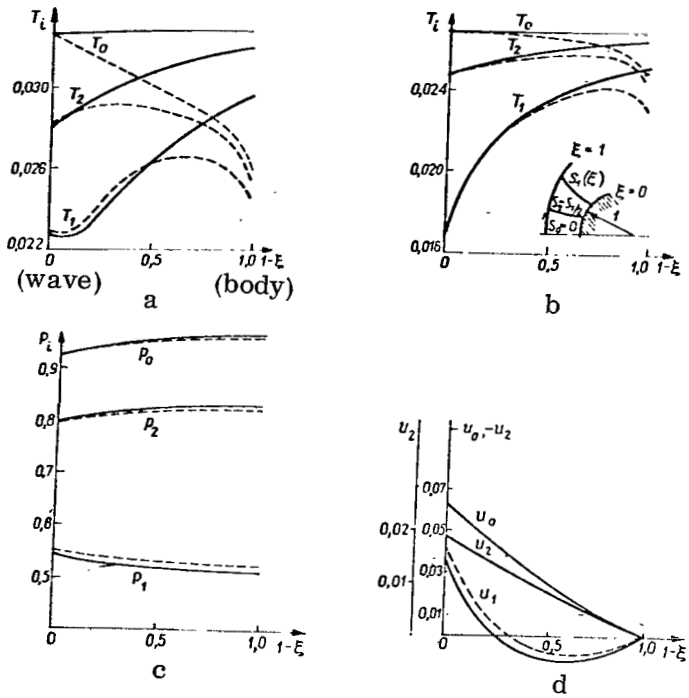
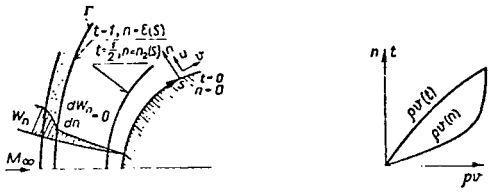


Figure 3.28.
—— - E = 0; ---- - E ≠ 0, $\bar{\bar{w}}\left(\nabla h - \frac{\nabla p}{\rho}\right) = -\frac{E-A}{\rho}$;
E = E(ρ, T) [5]; a) $M_\infty = 32$, $p_\infty = 0.01$ atm, $T_\infty = 250^\circ K$; b) $M_\infty = 34.5$, $p_\infty = 0.00025$ atm, $T_\infty = 250^\circ K$.

Let us briefly consider the statement of the problem. The orthogonal coordinate system s, n was used. For n = ∞ all the sought functions take on the values of variables in undisturbed flow. At a finite distance from the body one draws a sufficiently smooth line Γ ($n = \varepsilon(s)$), the location of which is tied to the region of the smeared out "shock wave" (Fig. 3.29a). Line Γ was selected in a manner such that regions with relatively sharp changes in functions, characteristic of the wave structure, be located outside the "shock layer" ($0 \leq n \leq \varepsilon(s)$) and be defined as a line at which $dw_n/dn = 0$. The Navier-Stokes equations were solved numerically for the "shock layer" region bounded by the body's contour and line Γ, while outside of it ($n \geq \varepsilon(s)$) these equations were replaced by similar, but simpler equations. Solution of these latter equations, satisfying the above conditions at infinity, permits one to find, with a certain accuracy, the boundary conditions for the equations in region $n \leq \varepsilon(s)$. Here, in accordance with the conservation laws, continuity conditions for the sought functions, as well as for derivatives of the enthalpy and velocity components normal to line Γ should be satisfied at this line. For the sake of simplification the Navier-Stokes operator for $n \geq \varepsilon(s)$ was represented in the form of a small-parameter expansion, with the parameter proportional to the curvature of line Γ. The calculated results presented here utilize only the zero term,



a b

Figure 3.29.

which represents the Becker system, describing local one-dimensional flow across the shock wave (the order of subsequent terms is $\sim O(1/\mathrm{Re})$ and higher).

Single integration of equations of the zero approximation yields boundary conditions at line Γ. These relations, together with the conditions at the body yield a sufficient number of boundary conditions for $n = 0$ and $n = \varepsilon(s)$ for solving the complete Navier-Stokes equations in the region $0 \leq n \leq \varepsilon(s)$ with the unknown boundary $n = \varepsilon(s)$.

A. I. Tolstykh [26, 40] considered, in the region of the "shock layer" equations obtained from the complete Navier-Stokes equations with the same order of accuracy as the boundary conditions (terms of the form $\frac{\partial^2 v}{\partial s^2}$, $\frac{\partial^2 h}{\partial s^2}$, $\frac{\partial^2 u}{\partial s \partial n} \sim O\left(\frac{1}{\mathrm{Re}}\right)$, etc., are missing). This system consists all the gasdynamic equations and equations of the boundary layer and possesses "elliptical" properties of upstream propagation of disturbances. The higher derivatives with respect to n in this system are the same as in the complete equations, hence the boundary conditions for $n = 0$ and $n = \varepsilon(s)$ remain as before.

The numerical algorithm of this system of equations was constructed by means of scheme I of the method of integral relations. The functions were approximated across the "shock layer" not with respect to coordinate n, but to the new variable t(n, s), which describes approximately the distributions of the tangential velocity component v for s = const. Upon changing the s, t coordinate system the variation of the function complexes being represented becomes smoother and approaches linearity (Fig. 3.29b). As in the case of an inviscid gas, here the number of boundary conditions of the approximating system for s = 0 was also found insufficient for starting integrating it (N variables are unkown), which is a result of the "elliptical" properties of the starting system of equations, which are due to the presence of partial derivatives of the pressure. It was found, however, that these unkowns are determined by N regularity conditions at singular points (N - 1 saddle points and one node point). /191

Figures 3.30-3.32 depict some results obtained by calculation for flow past a circular cylinder at $M_\infty = 5.2$ and Re = 12.5, 25 and 50 (the body was assumed to be thermally insulated, viscosity coefficient $\mu \sim h^{\varkappa}$, $\varkappa = 1.4$). Figure 3.30 displays distributions of velocity v in different sections s = const. It is seen that as one moves away from the axis of symmetry distributions become "fuller." Figure 3.31

gives the location of boundaries of the strips in the physical plane. Figure 3.32 presents the distributions of pressure $\bar{p}_0$, enthalpy h_0 and the variation in the friction coefficient c_f along the body's surface (c_f^* corresponds to free-molecular flow-diffusive reflection with accomodation coefficient of unity). As follows from the graph, the pressure distribution changes relatively little for different Re and is close to the distribution corresponding to the flow of inviscid gas (Re = ∞), while the variation of c_f is very sensitive to that of Re.

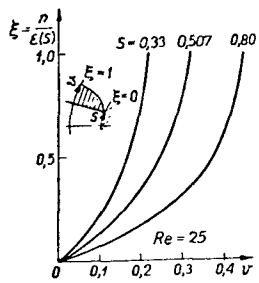


Figure 3.30.

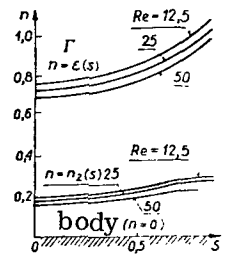


Figure 3.31.

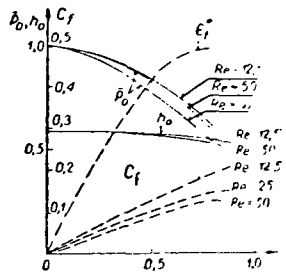


Figure 3.32.

M. M. Golomazov [26] used scheme II for developing a method for calculating the flow past axisymmetrical bodies at an angle of attack. The problem is solved with exact consideration of the region of influence, which is ensured by introducing the coordinate system $\eta = s/s_1(\xi, \vartheta)$; $\xi = n/\varepsilon(s, \theta)$; $\vartheta = \theta$, where $s = s_1(\xi, \vartheta)$ is the equation of the characteristic surface bounding the region of influence, $n = \varepsilon(s, \theta)$ /192
is the equation of the shock wave, θ is the angle in the transverse plane. The region of integration in the transverse plane ($0 \leq \vartheta \leq \pi$) is subdivided by rays $\vartheta = \text{const}$, and then trigonometric representation with respect to ϑ are used. Functions with respect to η are approximated in each meridional plane $\vartheta = \text{const}$, as this is done in scheme II (Fig. 3.33). Using the equations at the wave, body and compatibility equations at the characteristic surface we get a completely closed approximating system. Figure 3.34 shows calculations for perfect gas flows ($\varkappa = 1.4$) at an angle of attack ($\alpha = 5^\circ$) around ellipsoid of revolution ($\delta = 2$). The shape and location of the shock waves, as well as the pressure distribution along the strip boundaries are given for the different meridional sections.

Figures 3.35 and 3.36 compare results obtained by the method of integral relations for flow at zero angle of attack past a sphere and ellipsoids of revolution with $\delta = 0.5$ and 1.5 ($M_\infty = 3$, 4, 6.05, 8.06) with data obtained experimentally by G. M. Ryabinkov [26] ($\bar{p}_0$ is the pressure distribution along the body, referred to /195
the stagnation-point pressure, θ_{rad} is the angle of the polar coordinate system r, θ with a pole at the center of curvature of the nose) and V. G. Maslennikov and others (for ε_0).

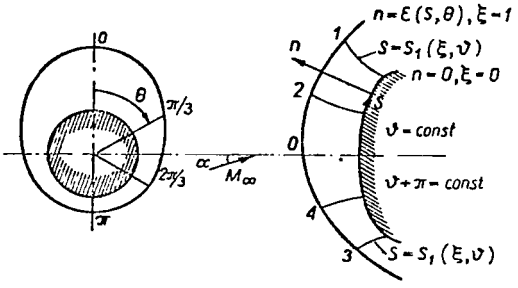


Figure 3.33.

$M_\infty = 10$, $\varkappa = 1.4$, $\delta = 2$, $\alpha \neq 0$.

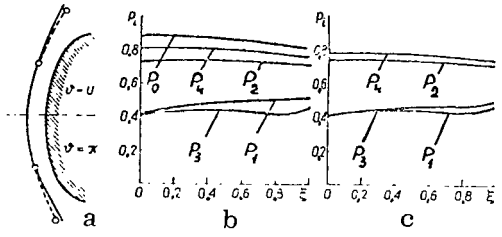


Figure 3.34.

$M_\infty = 10$, $\varkappa = 1.4$, $\delta = 2$; a) --- $\alpha = 0^\circ$, ∘∘∘ $\alpha = 1^\circ$; —— $\alpha = 5^\circ$; b) $\vartheta = 0, \pi$, $\alpha = 5^\circ$; c) $\vartheta = \pi/3, 2\pi/3$, $\alpha = 5^\circ$.

Figure 3.37 depicts the results of calculations for the flow past a sphere ($\varkappa = 1{,}4$) at low supersonic free-flow velocities ($M_\infty = 1.15$-2.0) obtained by F. D. Popov using scheme III of the method of integral relations. The shapes of shock waves and sonic lines are shown; experimental data obtained by A. P. Bedin and G. I. Mishin are also plotted there (triangles).

Different schemes of the method of integral relations, as well as results of numerous calculations of flow past blunt bodies are presented in tables and graphs in [26].

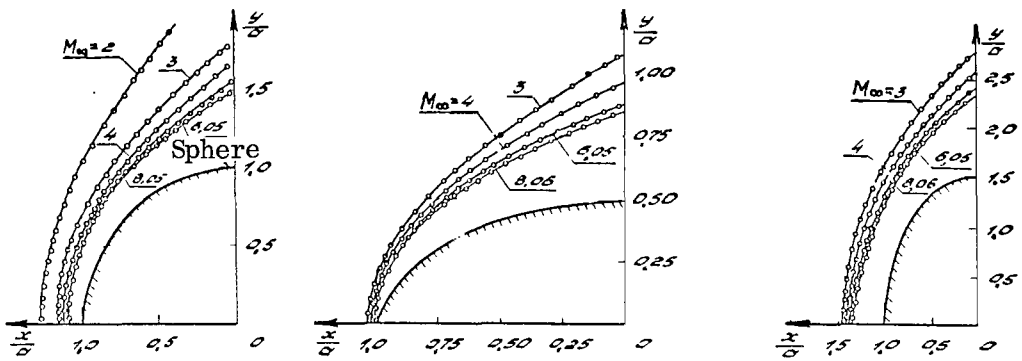


Figure 3.35.

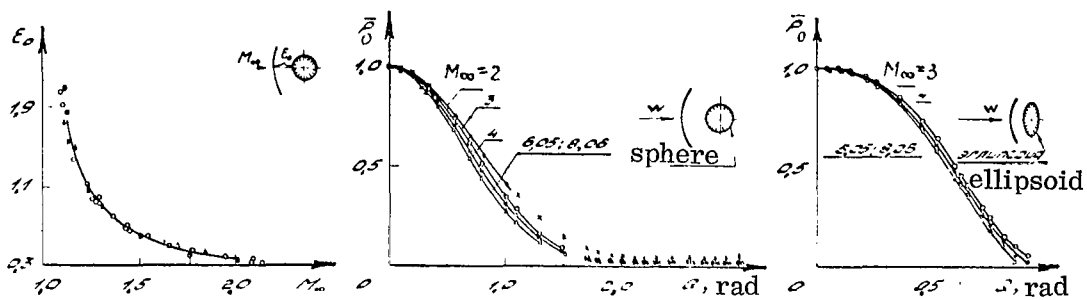


Figure 3.36

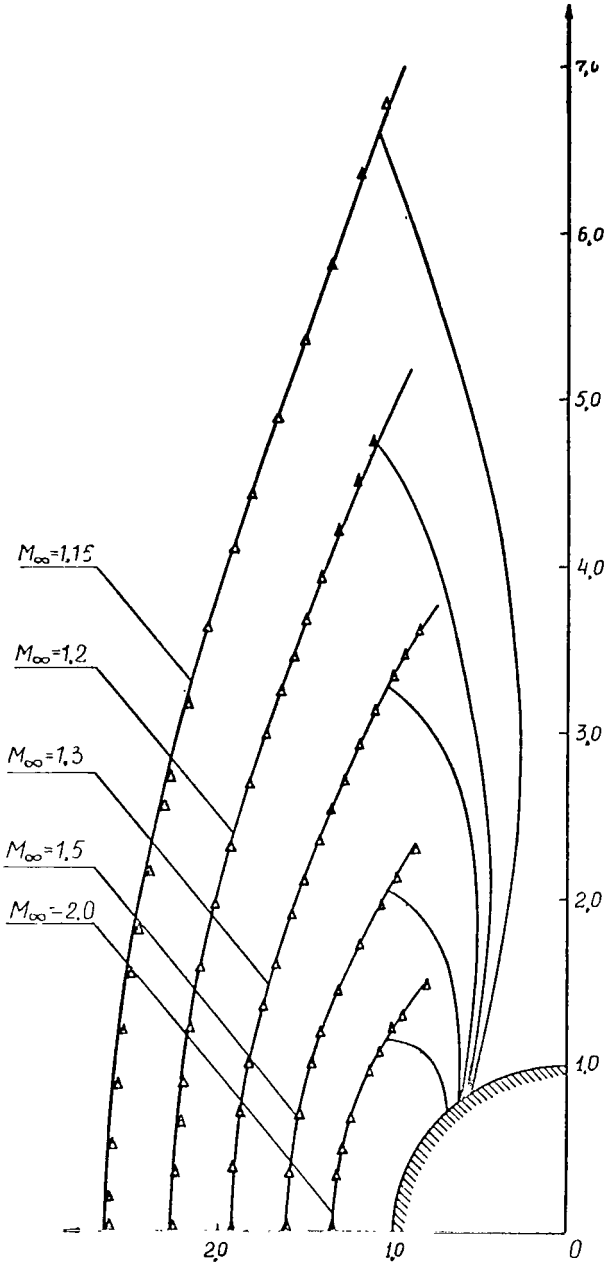

M_∞=1,15
M_∞=1,2
M_∞=1,3
M_∞=1,5
M_∞=2,0
7,0
6,0
5,0
4,0
3,0
2,0
1,0
0
2,0
1,0

Figure 3.37.

2. A Scheme of the Method of Straight Lines for Calculating Supersonic Flows Past Blunt Bodies With a Detached Shock Wave

G. F. Telenin [41, 42] suggested a numerical scheme of the method of straight lines for calculating mixed flows in the vicinity of analytically describable bodies. This scheme uses rectilinear subdivision of the region of integration, the latter being bounded from top by some ray (and not by a boundary characteristic) passing in the transonic zone. Calculations here are carried out by the "analytic" scheme, with computations for each specific case carried out in the elliptical region until the boundary ray passes into the transonic region and the data at it no longer have perceptible effect on the numerical solution. The use here of a rectilinear subdividing mesh for the region of integration makes it possible in general to simplify the form of the approximating operator. This scheme has been used for a large number of calculations for ideal gases, flows with physico-chemical transformations, as well as for three-dimensional flows (first done in 1964) [42].

1. We now present the computational algorithm, following [41]. We consider for simplicity axisymmetrical flow of a perfect gas ($\varkappa$ = const). The starting system of equations in dimensionless form can be written in the spherical coordinate system r, θ, φ, as follows:

$$\left.\begin{aligned}
&\left[\varkappa p^{(\varkappa-1)/\varkappa}\vartheta(\psi)-u^2\right]\frac{\partial u}{\partial r}-uv\frac{\partial v}{\partial r}+\frac{1}{r}\left\{\left[\varkappa p^{(\varkappa-1)/\varkappa}\vartheta(\psi)-\right.\right.\\
&\left.\left.-v^2\right]\frac{\partial v}{\partial r}-uv\frac{\partial u}{\partial\theta}\right\}+\frac{\varkappa\vartheta(\psi)}{r}p^{(\varkappa-1)/\varkappa}(2u+v\operatorname{ctg}\theta)=0;\\
&\frac{\partial v}{\partial r}=\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}-\frac{\varkappa}{\varkappa-1}r\sin\theta\frac{p}{\vartheta(\psi)}\frac{d\vartheta(\psi)}{d\psi};\\
&\frac{\partial p}{\partial r}=\frac{1}{\vartheta(\psi)}p^{1/\varkappa}\left(\frac{v^2}{r}-\frac{v}{r}\frac{\partial u}{\partial\theta}-u\frac{\partial u}{\partial r}\right);\quad \frac{\partial\psi}{\partial r}=\frac{1}{\vartheta(\psi)}p^{1/\varkappa}rv\sin\theta\\
&\qquad\left(\vartheta(\psi)=\frac{1}{\rho}p^{1/\varkappa}\right),
\end{aligned}\right\}\qquad(3.14)$$

/15

where u and v are projections of velocity vector w onto unit vectors $\bar{r}$ and $\bar{\theta}$, referred to w_{max}, p and ρ are the pressure and density referred to $\rho_\infty w_{max}^2$ and ρ_∞, respectively; r is referred to characteristic dimension L, while the streamline ψ is referred to $\rho_\infty w_{max} L^2$.

As usual, the boundary conditions are specified at the shock wave, axis of symmetry and the body's contour. The solution is sought in a class of functions with a continuous limited derivative for analytically describable bodies.

The boundary-value problem is more conveniently solved by replacing r, θ by the coordinate $\xi = \frac{r-r_0}{\varepsilon}$, θ, where $r = r_0(\theta)$ is the equation of the body's contour,

while $r = r_1(\theta) = r_0(\theta) + \varepsilon(\theta)$ is the equation of the shock wave. In variables ξ, θ the region of integration between the shock wave and the body is transformed into the strip $0 \leq \xi \leq 1$. /197

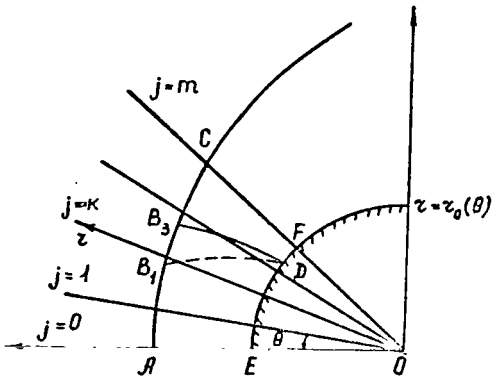


Figure 3.38.

We draw a ray $\theta = \text{const}$ in the upper half plane m + 1 (Fig. 3.38) and represent the sought functions by means of Legendre polynomials, having reference to symmetry of the flow with respect to $\theta = 0$. Then we get for functions

$$u = \sum_{j=0}^{m} u_j^0(\xi)\theta^{2j}; \quad v = \sum_{j=0}^{m} v_j^0(\xi)\theta^{2j+1}; \quad r_1 = \sum_{j=1}^{m} r_j^0\theta^{2j} \tag{3.15}$$

and for derivatives with respect to θ

$$\left.\begin{aligned} u' &= \sum_{j=1}^{m} 2ju_j^0(\xi)\theta^{2j-1}, \quad v' = \sum_{j=0}^{m} (2j+1)v_j^0(\xi)\theta^{2j}; \\ r_1' &= \sum_{j=0}^{m} 2jr_j^0\theta^{2j-1}. \end{aligned}\right\} \tag{3.16}$$

Quantities u_j^0, v_j^0 and r_1^0 are linear functions of the values of u, v and r_1 at the m + 1 rays in the upper half plane.

Substituting the expressions for derivatives with respect to θ in Eqs. (3.14) and requiring that the obtained expressions be satisfied identically at each ray, we get an approximating system of ordinary differential equations for u_k, v_k, p_k and ψ_k,

which are the approximate values of these functions at the rays. Solving this equation for derivatives with respect to ξ, we finally get

$$\frac{du_k}{d\xi} = \frac{\varepsilon_k(r_{0k}+\xi\varepsilon_k)}{\Delta_k}\Big\{c_k[u_k v_k(r_{0k}+\xi\varepsilon_k)^2 + b_k(r'_{0k}+\xi\varepsilon'_k)\times$$

$$(r_{0k}+\xi\varepsilon_k)] - d_k(2u_k+v_k\operatorname{ctg}\theta_k) - b_k v'_k + (u'_k - v'_k)\times$$

$$\frac{b_k(r'_{0k}+\xi\varepsilon'_k)}{r_{0k}+\xi\varepsilon_k} + (2u'_k - v_k)u_k v_k\Big\};$$

$$\frac{dv_k}{d\xi} = \varepsilon_k\left[\frac{1}{r_{0k}+\xi\varepsilon_k}\left(u'_k - \frac{r'_{0k}+\xi\varepsilon'_k}{\varepsilon_k}\frac{du_k}{d\xi}\right) - \frac{v_k}{r_{0k}+\xi\varepsilon_k} + c_k(r_{0k}+\xi\varepsilon_k)\right];$$

$$\frac{dp_k}{d\xi} = \frac{\varepsilon_k p^{1/\varkappa}}{\vartheta(\psi_k)}\left\{\frac{v_k^2}{r_{0k}+\xi\varepsilon_k} - \frac{v_k}{r_{0k}+\xi\varepsilon_k}\left[u'_k - \frac{r'_{0k}+\xi\varepsilon'_k}{\varepsilon_k}\frac{du_k}{d\xi}\right] - \frac{u_k}{\varepsilon_k}\frac{du_k}{d\xi}\right\};$$

$$\frac{d\psi_k}{d\xi} = \varepsilon_k(r_{0k}+\xi\varepsilon_k)\frac{1}{\vartheta(\psi_k)}p^{1/\varkappa}v_k\sin\theta_k;$$

$$\rho_k = \left[\frac{p_k}{\vartheta(\psi_k)}\right]^{1/\varkappa} \qquad (k = 0, 1, \ldots, m),$$

where

$$a_k = \varkappa p^{\varkappa-1}\vartheta(\psi_k) - u_k^2; \qquad b_k = \varkappa p^{(\varkappa-1)/\varkappa}\vartheta(\psi_k) - \vartheta_k^2;$$

$$c_k = -\sin\theta_k\frac{\varkappa}{\varkappa-1}\frac{p_k}{\vartheta(\psi_k)}\frac{d\vartheta(\psi)}{d\psi}\Big|_{\psi=\psi_k}; \quad d_k = \varkappa p_k^{(\varkappa-1)/\varkappa}\vartheta(\psi_k),$$

with subscript k denoting the values of variables at the kth ray, the prime denoting derivatives with respect to θ, calculated from Eqs. (3.16).

At the zero ray (k = 0) all the terms of the second and fourth of equations (3.15) vanish identically. To improve the system's accuracy it is expedient to introduce here equations for $\left(\frac{\partial v}{\partial\theta}\right)_{\theta=0} = v_0^0$.

The approximating system should be supplemented by an equation relating the detachment ε of the shock wave from the body to the angle the shock wave makes with the axis of symmetry, and the boundary conditions at the wave and body serve as boundary conditions for the system of ordinary differential equations.

The algorithm of the numerical solution reduces to the following. Specification of m + 1 variables r_j^0 (j = 0, 1, ..., m) determines approximately the equation of the shock wave, while the use of relationships at the shock wave makes it possible to determine all of the flow variables behind the shock wave. Solving then the Cauchy problem for the approximating system, we shall determine the values of variables in nodes at the surface of the body which, in general, do not satisfy the boundary condition on the body. Selecting by iteration values of r_j^0 in such a manner that the boundary condition at the body be satisfied in all the nodes on the latter's surface

with the required accuracy, we shall obtain the sought solution of the approximating system in the mth approximation.

The computational method thus consists in constructing a sequence of equations which satisfies in the limit all the boundary conditions in region ACFE: relations on the shock AC, symmetry conditions on the axis segment AE, and the boundary condition on the body's surface EF. /199

The additional condition of bounded derivatives (nonvanishing of the Jacobian $\partial(x, y)/\partial(w_x w_y)$) is satisfied automatically, since each term of the sequence of approximate solution satisfies it. If the shape of the shock for which a limiting line arises in region ACEF is initially specified or is obtained in the process of iteration, it will be automatically set aside by the program, since the process of computation would be interrupted. The sought solution and the solutions obtained in the course of iterations belong to the class of analytic functions for which the Cauchy problem is correct.

2. Results calculated using this scheme are now presented [41]. Figures 3.39-3.42 illustrate the effect of M_∞ and of the shape of the body on the geometric pattern (relative location of the body's surface, shock wave and sonic line) of flow of a perfect gas with $\varkappa = 1.4$. Figure 3.39 shows the effect of changes in M_∞ in a wide range for flow past an ellipsoid with ratio of semiaxes $\delta = 2$. Figure 3.40 depicts the flow pattern at $M_\infty = 3$ for a family of bodies with the nose-section shape defined by the expression $x^n + y^n = 1$. For $n > 2$ the radius of curvature R_0 at the stagnation point is equal to infinity, while when n goes from 2 to 10 the ratio of the minimum (in the vicinity of the corner) radius of curvature R_{min} to the body's diameter (R_{min}/D) changes from 0.5 to 0.07. Figure 3.41, pertaining to flow at $M_\infty = 3$ past Cassini's ovals, the contour of which is given by the expression $(x^2 + y^2)^2 + 2c^2(x^2 - y^2) = a^4 - c^4$ $(a^2 + c^2 = 1)$, shows the effect of a contour concavity in the vicinity of the stagnation point. Figure 3.42 shows the flow at $M_\infty = 3$ past body "a", whose contour has a break in curvature at point c and is formed by conjugated circles with radii $R_0/D = 1$ in the vicinity of the stagnation point and $R_1/D = 0.2$ in the vicinity of the middle section, and past body "b", consisting of a 60° segment with a corner point. Figure 3.43 depicts the pressure distribution on the surfaces of bodies of various shapes in a flow with $M_\infty = 3$. To prevent the curves from merging into one, the polar angle is used as the argument along the body's surface (for body "b" $\theta = \theta_1$, where θ_1 is the angle made with the pole at the center of the circle with radius R_0 (see Fig. 3.41), for body "a" $\theta = \theta_1$ up to point c, and then $\theta = \theta_{1c} + \theta_2$ (see Fig. 3.42). The points represent experimental data of Yu. Ya. Karpeyskiy. /201

The supersonic flow of nonequilibrium air past a sphere was studied with allowance for the kinetics of excitation of vibrations in O_2 and N_2, dissociation and transfer processes, including $N_2 + O_2 = 2NO$ and ionization by atom collisions $N + O = NO^+ + e$,

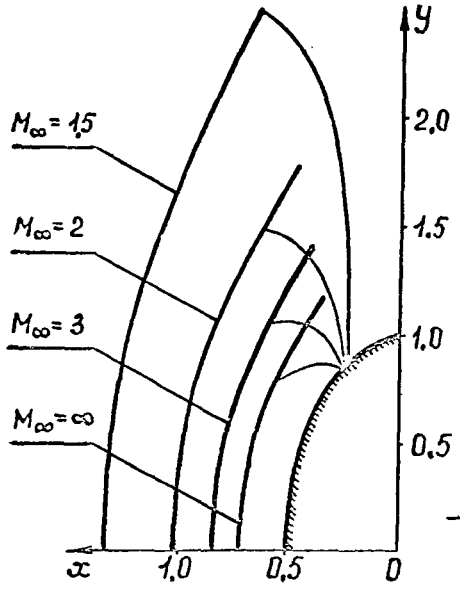


Figure 3.39.

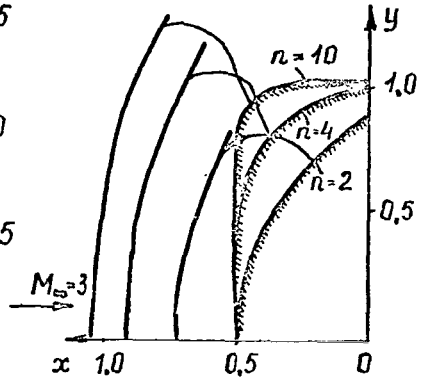


Figure 3.40.

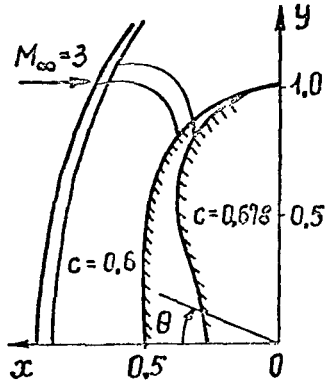


Figure 3.41.

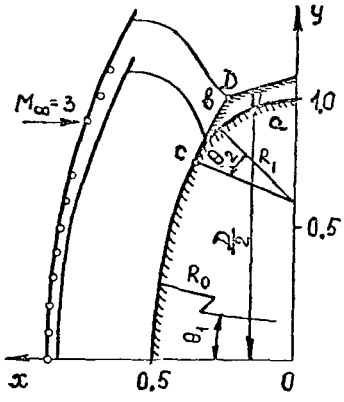


Figure 3.42.

$N + N = N_2^+ + e$, and $O + O = O_2^+ + e$, which satisfactorily describe the process up to $M \approx 30$. The effect of physico-chemical transformations occurring behind the shock wave on its detachment is shown in Fig. 3.44, where pressure p is given in bars, curves 1 and 2 correspond to the equilibrium, curve 3 corresponding to frozen, and the remaining curves to nonequilibrium external flow.

The calculations represented in Figs. 3.39-3.41 were carried out with a 9-point scheme ($m + 1 = 9$) and those in Fig. 3.42 with a 5-point scheme ($m + 1 = 5$). Without

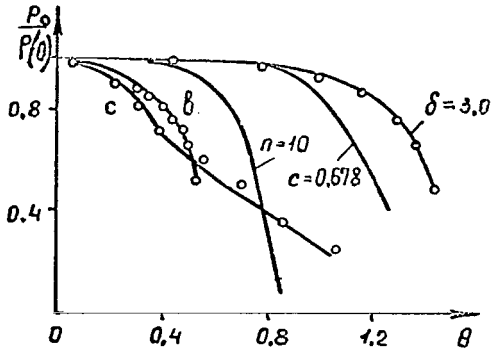


Figure 3.43.

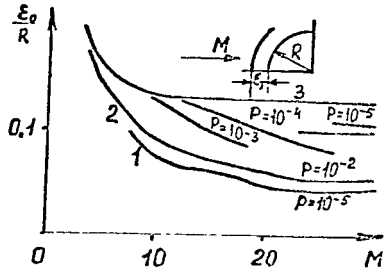


Figure 3.44.

$T_\infty = 250^\circ K$, $R_0 = 0.015$ m.

dwelling here on the determined degree of accuracy, we note that the total error in the computation of most of the versions did not exceed 1%. A review of studies and results obtained using this scheme is presented in [43].

3. On the Use of Pseudo-Viscosity in Numerical Solution of the Inverse Problem of Gasdynamics

Alongside with studies concerned with solving the direct problem of supersonic flow about a blunt body, a large number of investigations have appeared during the past several years concerning the so-called inverse problem. In methods of this type specification of the shock wave defines all the initial conditions, which makes it possible to construct a solution by moving step by step away from the wave. This somewhat simplifies the calculations; however, it requires assuming that the shape of the blunt body is that which is obtained from the calculations. In cases when a solution must be found for a body of specified shape, trial and error must be used. The required shape is obtained approximately to a given degree of accuracy, which is determined by the number of variables of the shock-wave family. /202

Another substantial shortcoming of methods of the inverse type is their inherent instability, brought about by the fact that the inverse problem of gasdynamics (which reduces to the Cauchy problem for a system of elliptical-type differential equations) is incorrectly stated.

The main difficulties in numerical solution of the inverse problem here consist in the fact that the errors due to approximating and rounding increase exponentially as one moves away from the shock wave. The solution becomes "spoiled" as early as after 10-12 steps. The difference grid which is obtained with the above limit on the number of steps may be found to be exceedingly coarse in many important cases, for example, in calculating the flow past blunt bodies with allowance for relaxation in the shock layer.

This instability can be overcome only with extreme difficulty. Thus, in the Van Dyke and Mangler methods this instability requires taking a small number of steps when computing from the shock wave to the body, which appreciably reduces the computational accuracy. In the Garabedian-Libershteyn method this instability is eliminated by analytically extending the initial conditions into the region of a fictitious third dimension, where the equations are parabolic and the solution is constructed by the method of characteristics. Systematic computations using this method can be carried out only with great difficulties.

Practical needs have resulted in the appearance of various methods for regularizing such problems [44, 45]. M. M. Lavrent'yev [44] presented a method for solving the initial-conditions problem for the Laplace equation by introducing higher than second-order derivatives (with small coefficient) of the unknown function into the equation.

L. I. Severinov developed a method for regularizing the inverse problem of gasdynamics by introducing into the differential equations of gasdynamics higher-order derivatives (with small coefficients) of the sought functions (pseudo-viscosity). This "damps" out the instability in computing the problem with initial conditions at the shock wave, and stable computations are possible. Following Severinov [46] we now present the algorithm developed for his method. /20

1. We consider supersonic flow past a blunt body of revolution at zero angle of attack. The detached shock wave is specified in the form $z_1/R_0 = \mathrm{ch}\left(\frac{r_1}{R_0} - 1\right)$, where r_1 and z_1 are cylindrical coordinates in the physical plane, with z_1 measured along the flow from the point of intersection of the shock front with the axis of symmetry, R_0 is the radius of curvature of the shock wave at $r_1 = 0$. The shock wave was selected in such a manner as to obtain a close-to-spherical body.

Let x and y be coordinates in an orthogonal curvilinear system, traveling with the shock wave. Lines x = const are perpendicular to the shock front, lines y = const are at equal distances from it (at the shock wave y = 0). The Lamé coefficients in this coordinate system have the form $H_x = 1 - y/(1 + x^2)$, $H_y = 1$. We convert to the cylindrical coordinate system using the expressions

$$r = \frac{r_1}{R_0} = \ln(x+\sqrt{1+x^2}) - \frac{xy}{\sqrt{1+x^2}}, \quad z = \frac{z_1}{R_0} = \sqrt{1+x^2} - 1 + \frac{y}{1+x^2}.$$

We introduce second derivatives, with respect to x, of the velocity vector components into the differential equations of gasdynamics. For this we consider a gas which has a stress tensor with the components

$$p_{xx} = -p + \frac{\varepsilon_1}{H_x}\left[\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_0\right]; \quad p_{xy} = \frac{\varepsilon_2 \partial v}{H_x \partial x}; \quad p_{yx} = 0; \quad p_{yy} = -p.$$

where u and v are the velocity vector components in the x and y directions, respectively, p is the pressure, ε_1 and ε_2 are small constant coefficients and $\left(\frac{\partial u}{\partial x}\right)_0$ is the derivative at x = 0 (function of y).

Then the system of equations of gasdynamics (equations of motion, conservation of the mass and energy) will be written in the form

/204

$$\frac{\rho u}{H_x}\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}-\frac{\rho uv}{(1+x^2)H_x}=-\frac{1}{H_x}\frac{\partial p}{\partial x}+\frac{\varepsilon_1}{H_x^2}\frac{\partial^2 u}{\partial x^2}-\frac{\varepsilon_2}{(1+x^2)H_x^2}\frac{\partial v}{\partial x}+$$

$$\frac{\varepsilon_1}{rH_x}\left[\frac{\partial u}{\partial x}-\left(\frac{\partial u}{\partial x}\right)_0\right]\left(\frac{1}{\sqrt{1+x^2}}-\frac{2rxy}{(1+x^2)H_x^2}\right);$$

$$\frac{\rho u}{H_x}\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\frac{\rho u^2}{(1+x^2)H_x}=-\frac{\partial p}{\partial y}+\frac{\varepsilon_1\partial^2 v}{H_x^2\partial x^2}+\frac{\varepsilon_1}{(1+x^2)H_x^2}\left[\frac{\partial u}{\partial x}-\left(\frac{\partial u}{\partial x}\right)_0\right]+$$

$$\frac{\varepsilon_2}{rH_x}\frac{\partial v}{\partial x}\left(\frac{1}{\sqrt{1+x^2}}-\frac{2rxy}{(1+x^2)H_x^2}\right);$$

$$v\frac{\partial\rho}{\partial y}+\rho\frac{\partial v}{\partial y}+\frac{u}{H_x}\frac{\partial\rho}{\partial x}+\frac{\rho}{H_x}\frac{\partial u}{\partial x}+\frac{\rho u}{r\sqrt{1+x^2}}-\rho v\left(\frac{1}{(1+x^2)H_x}+\frac{x}{r\sqrt{1+x^2}}\right)=0;$$

$$\frac{u}{H_x}\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}-\frac{u}{\rho H_x}\frac{\partial p}{\partial x}-\frac{v}{\rho}\frac{\partial p}{\partial y}=\frac{\varepsilon_1}{\rho H_x^2}\frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial x}-\left(\frac{\partial u}{\partial x}\right)_0\right]+$$

$$\frac{\varepsilon_2}{\rho H_x^2}\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{u}{1+x^2}\right),$$

where h is the specific enthalpy, ρ is the density, and $r = r_1/R_0$. This system is joined by equations of state and equations describing relaxation processes.

The calculations described above were carried out for a system of chemical reactions, coinciding with the Duff-Davidson system. Allowance was also made for nonequilibrium excitation of vibrational degrees of freedom of nitrogen and oxygen molecules. The calculations were made for a medium at rest, consisting of a mixture of argon and molecular oxygen and nitrogen, the mole fractions being 0.0097, 0.2095 and 0.7808, respectively.

2. The difference net in the x, y plane had rectangular cells. The derivatives of the sought functions contained in the gasdynamics equations were approximated by intersections marked by crosses in Fig. 3.45a. Derivatives with respect to x at the nth strip were approximated within $(\Delta x)^4$. There was no need to use schemes with a higher degree of accuracy with respect to x. Successive approximations

were used in zones with large gradients. The derivatives of the concentrations were approximated by grid points marked in Fig. 3.45b.

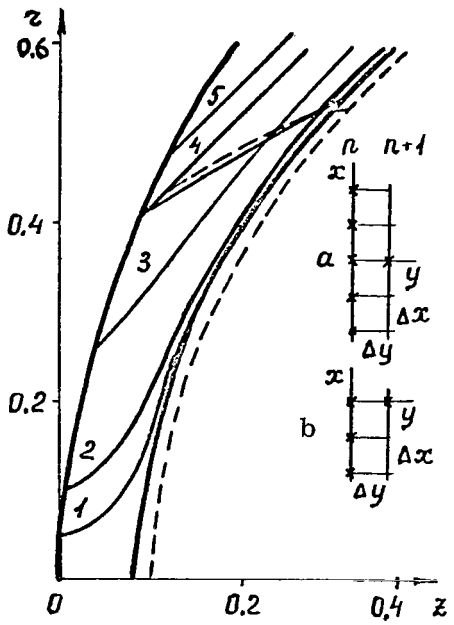


Figure 3.45.

Figures 3.46-3.48 present some calculated results, which show the effects of instability in numerical computation of the inverse problem of gasdynamics and also the smoothing effect of pseudo-viscosity. The free-flow variables were $p_\infty = 1$ kgf/m^2, $T_\infty = 230^\circ$K, $M_\infty = 12.0$, and $R_0 = 1$ m.

The broken lines in Figs. 3.46 and 3.47 represent second differences of u and v, calculated along the strip y = const at the 17th step from the shock wave for $\varepsilon_1 = \varepsilon_2 = 0$, i.e., without second-order derivatives in the gasdynamics equations. It was found from experience that the magnitude of the stagnation enthalpy is very sensitive to computational errors, and hence the problem was solved using the differential equation of conservation of energy, while the requirement that the stagnation enthalpy be constant was checked, in order to verify computational accuracy, at each intersection on the grid after all the variables were calculated. The broken line in Fig. 3.48 represents the stagnation enthalpy at the body's contour at $\varepsilon_1 = \varepsilon_2 = 0$. As follows from this figure, the variation in the stagnation enthalpy along the contour is excessively high. In addition, the contour obtained is a broken line and the impermeability condition is not satisfied at it.

The smooth curves in Figs. 3.46 and 3.47, drawn along points obtained with $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = -10^{-2}$, illustrate the smoothing effect of pseudo-viscosity. The

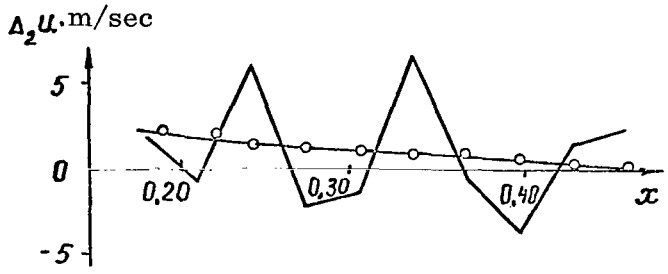


Figure 3.46.

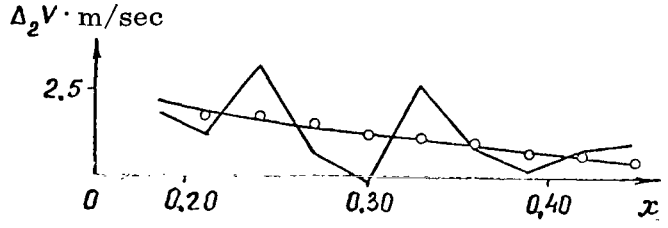


Figure 3.47.

/207

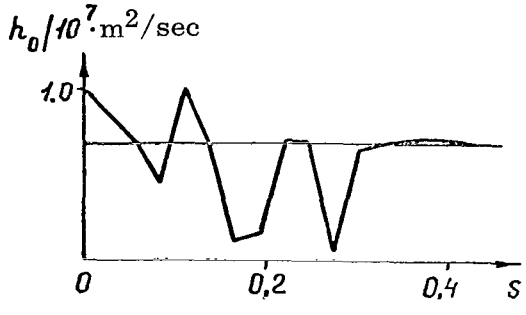


Figure 3.48.

stagnation enthalpy with the above values of ε_1 and ε_2 along the body's contour and in the region of flow computation differed from the stagnation enthalpy of undisturbed flow by not more than 0.3%. The calculated points lie in Fig. 3.48 on the straight line h_0 = const, which shows that the computations are sufficiently accurate. In this example the minimum number of steps from the wave to the body was 16. Proper selection of magnitudes of ε_1 and ε_2 always made it possible to obtain a smooth body contour and to satisfy the impermeability conditions at it.

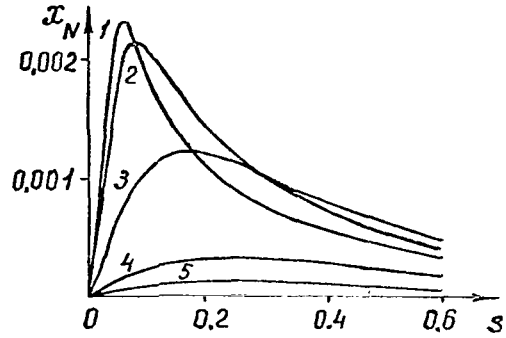


Figure 3.49.

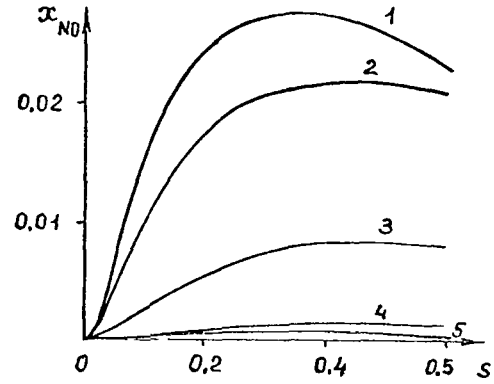


Figure 3.50.

The step for y was initially selected in such a manner as to provide for required accuracy and stable computation of concentrations, after which values of ε_1 and ε_2 were selected for eliminating the effect of the rapid increase in errors inherent to the problem with initial conditions for a system of elliptical-type equations. It was possible to bring the number of steps between the shock front and body to 80-90. The /2 calculations were carried out in a minimal region of influence.

3. Figures 3.45 and 3.49-3.52 illustrate results of computations with p_∞ = 0.595 kgf/m^2, $T_\infty = 218^\circ$K, $M_\infty = 11.8$, $R_0 = 1$ m. Figure 3.45 shows the body's contour, shock wave, sonic line and streamlines. A dashed line on this figure also shows the contour of the body and the sonic line of the flow "frozen" in the shock front. Figures 3.49-3.52 present the mole fractions of atomic nitrogen, nitrogen oxide and the energy of vibrational degrees of molecular freedom for oxygen and nitrogen along streamlines drawn in Fig. 3.45. In all the figures s is the length along the body's contour or streamlines, referred to R_0, $z = z_1/R_0$. Values of s = 0

in Fig. 3.48 correspond to the stagnation point. In the remaining figures s = 0 at the shock wave. In this example $\Delta x = 0.03$, $\Delta y = 0.0025$, $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = -10^{-2}$.

/208

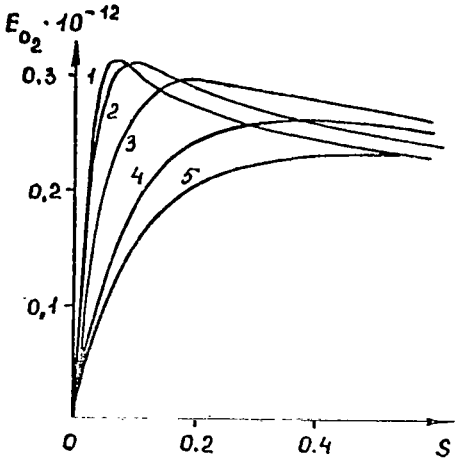


Figure 3.51.

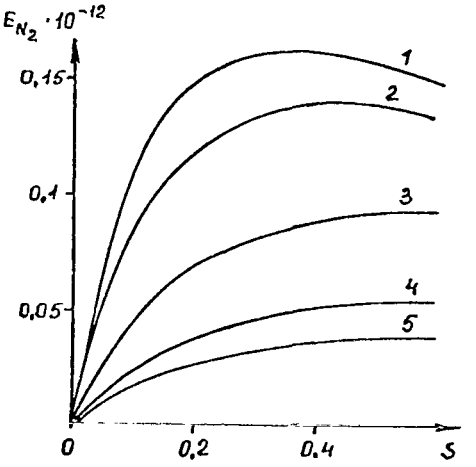


Figure 3.52.

It can be expected, on the basis of the calculations above, that the use of higher derivatives will be beneficial also in numerical solution of other, similar problems.

The methods for doing so can apparently be quite numerous, depending on the conditions of the problem.

Pseudo-viscosity was previously introduced into gasdynamics equations in order to enable continuous computations across shock waves in numerical solution of problems. The starting system of differential equations in these cases was of the hyperbolic type. The computations described above present a second use of pseudo-viscosity.

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CERTAIN NUMERICAL METHODS FOR SOLVING EQUATIONS OF THE BOUNDARY LAYER

V. V. Shchennikov

All the available numerical methods for solving boundary-layer equations can in substance be subdivided into two classes: a) methods based on the use of integral relations, which can be obtained from differential equations; b) finite-difference methods, based on difference representation of starting differential equations and boundary conditions. /214

The first class of methods includes the well-known Pohlhausen method, the Loytsyanskiy-Dorodnitsyn parametric methods and, finally, the method of integral relations (simple and generalized) due to Dorodnitsyn, which has lately come into extensive use.

An item of importance in considering finite difference methods in solving boundary-layer equations is the initial form of these equations. The known von Mises transformation transforms the equation of an incompressible boundary layer to the heat-conduction equation:

$$\frac{\partial z}{\partial x} = \nu\sqrt{V^2 - z}\,\frac{\partial^2 z}{\partial \psi^2},$$

where $z = V^2 - u$ is the energy defect.

The convenience of this equation is apparently the possibility of using well-developed methods for solving the heat conduction equation. This approach has the shortcoming that the coefficient of the leading derivative vanishes at the boundary of the solid body ($\sqrt{V^2 - z} = 0$ for $\psi = 0$). A way out of this situation is to isolate the singularity in the vicinity of the body. It is found that the solution in the vicinity of the wall has the form /215

$$u \cong a + bx + cx^{3/2} + dx^2 + 0(x^2),$$

where $u = (1 - x)z$; $x = \frac{\psi}{\psi_0 + \psi}$ (a_m, b_m, c_m – const). Approximating the solution by means of the polynomials

$$u \approx a_m + b_m x + c_m x^{3/2},$$

we get a difference representation of the equation, which is then solved by one of the methods.

Another transformation of boundary-layer equations is that due to Crocco. The equation in Crocco variables has the form

$$\frac{\partial^2 \tau}{\partial u^2} + u \frac{\partial}{\partial x}\left(\frac{\rho u}{\tau}\right) - \frac{dp}{dx} \cdot \frac{\partial}{\partial u}\left(\frac{\mu}{\tau}\right) = 0,$$

where $\tau = \mu \frac{\partial u}{\partial x}$. The advantage of this transformation is obvious and consists in the fact that the domains of definition of the solution ($0 \leq u \leq 1$) are finite. Clearly, this transformation requires the existence of the single-valued function $y = y(u)$. It is obvious that accelerated flows in the case of a heated wall (when the longitudinal velocity distributions in the boundary layer have the form shown in Figs. 1 and 2) cannot be described in terms of Crocco variables. In the majority of cases difference schemes are constructed with boundary-layer equations in terms of ordinary physical variables (velocity, density, pressure).

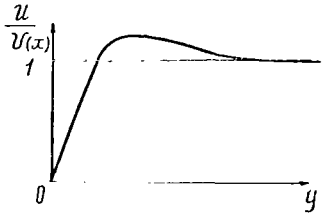

Figure 1.

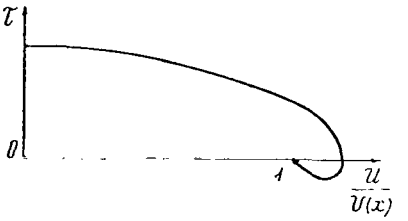

Figure 2.

We shall consider two of the most interesting of the above methods for solving boundary-layer equations: the A. A. Dorodnitsyn method of integral relations and one of the difference methods.

1. The Generalized Method of Integral Relations

/216

This method was presented in general outline in O. M. Belotserkovskiy's lectures. We shall illustrate it in more detail in its application to equations of a laminar, incompressible boundary layer. The starting equations have the form

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= VV' + \nu\frac{\partial^2 u}{\partial y^2}; \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad (1.0)$$

with the boundary conditions

$$u = v = 0 \quad \text{for} \quad y = 0;$$
$$u \to V \quad \text{for} \quad y \to \infty,$$

where V = V(x).

Using the Dorodnitsyn transformation

$$\bar{u} = \frac{u}{V}; \quad \bar{v} = \frac{v}{V\sqrt{\nu}}; \quad \xi = \int_0^x V dx; \quad \eta = \frac{1}{\sqrt{\nu}} \int_0^y V d\eta = \frac{Vy}{\sqrt{\nu}},$$

Eqs. (1.0) are reduced to the form

$$\left.\begin{aligned} \bar{u}\frac{\partial \bar{u}}{\partial \xi} + w\frac{\partial \bar{u}}{\partial \eta} &= \frac{\dot{V}}{V}(1-\bar{u}^2) + \frac{\partial^2 \bar{u}}{\partial \eta^2}; \\ \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial w}{\partial \eta} &= 0, \end{aligned}\right\} \qquad (1.1)$$

where

$$w = \bar{v} + \frac{\dot{V}}{V}\eta\bar{u},$$

with boundary conditions

$$\bar{u} = w = 0 \quad \text{for} \quad \eta = 0;$$
$$\bar{u} \to 1 \quad \text{for} \quad \eta \to \infty.$$

Subsequently we will have to know the behavior of this solution in the starting segment of the boundary layer. It is known that, if $V = cx^m$, then the solution can be represented in the form

$$\bar{u} = f_0(yx^{1/2(m-1)}) + \ldots$$

or in the ξ, η coordinates

$$\bar{u} \sim f_0\left(\text{const}\,\frac{\eta}{\sqrt{\xi}}\right) = \varphi_0\left(\frac{\eta}{\sqrt{\xi}}\right)$$

for any m. Seeking a solution in the form /217

$$\bar{u} = \varphi'\left(\frac{\eta}{\sqrt{\xi}}\right)$$

we get for function φ the known Falkner-Skan equation

$$\varphi''' + \varphi\varphi'' + \frac{2m}{m+1}(1 - \varphi'^2) = 0$$

with the condition

$$\varphi(0) = \varphi'(0) = 0; \quad \varphi'(\infty) = 1.$$

We note that in this case the stress at the wall is expressed, in terms of function φ, by

$$\tau_0 = \rho\sqrt{\nu}\, V^2\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0} = \rho V^2 \sqrt{\nu}\, \frac{\varphi''(0)}{\sqrt{2\xi}}.$$

To obtain the integral relation we multiply the second of equations (1.1) by $f(\bar{u})$, and the first of equations (1.1) by $f'(\bar{u})$ and then add the expressions thus obtained. As a result we get (omitting the bar over u)

$$\frac{\partial}{\partial \xi}[uf(u)] + \frac{\partial}{\partial \eta}[wf(u)] = \frac{\dot{V}}{V} f'(u)(1 - u^2) + f'(u)\frac{\partial^2 u}{\partial \eta^2}.$$

All that is required of function f(u) is that it tend sufficiently rapidly to 0 as $\eta \to \infty$.

Integrating the equation thus obtained with respect to η from 0 to ∞, we get

$$\frac{d}{d\xi}\int_0^\infty uf(u)\,d\eta = \frac{\dot{V}}{V}\int_0^\infty (1-u^2)f'(u)\,du - f'(0)\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} - \int_0^\infty \left(\frac{\partial u}{\partial \eta}\right)^2 f''(u)\,d\eta.$$

We introduce the notation

$$\theta = \frac{1}{\partial u/\partial \eta} \quad \text{and} \quad \theta_0 = \left(\frac{1}{\partial u/\partial \eta}\right)_{\eta=0}.$$

which finally yields the following integral relation

$$\frac{d}{d\xi}\int_0^1 \theta u f_i(u)du = \frac{\dot{V}}{V}\int_0^1 \theta(1-u^2)f'(u)du - \frac{f'(0)}{\theta_0} - \int_0^1 \frac{f''(u)}{\theta}du. \qquad (1.2)$$

In particular, for f(u) = 1 - u we have an analog of the known Karman integral relation for incompressible laminar boundary layers. /2

Now we must select function $\{f_n\}_N$. In order to do this, we utilize the fact that function θ in the boundary layer has the qualitative behavior:

$$\theta \sim 0\left(\frac{1}{1-u}\right) \quad \text{as} \quad u \to 1.$$

With reference to the above, we represent θ in the form of the following power polynomial in U:

$$\theta = \frac{1}{1-u}(a_0 + a_1 u + \ldots + a_{N-1}u^{N-1}),$$

similarly also for $1/\theta$:

$$\frac{1}{\theta} = (1-u)(b_0 + b_1 u + \ldots + b_{N-1}u^{N-1}).$$

Coefficients a_0 and b_0 are functions of the values of θ and $1/\theta$ at interpolation points

$$u_n = \frac{n}{N} \qquad (n = 0, 1, 2, \ldots, N-1).$$

It is natural to select functions f_n in the form

$$f_n(x) = (1-u)^n.$$

This ensures convergence of all the integrals in integral relation (1.2). We shall present the final form of approximating equations for the first three approximations. It will be useful to note that these equations are universal and are functions of only one variable typifying the body's geometry, namely the ratio $\dot{V}/V$ [1].

The system of the first approximation. The approximating expressions are

$$\theta = \frac{\theta_0}{1-u}; \quad \frac{1}{\theta} = \frac{1}{\theta_0}(1-u).$$

The differential equation has the form

$$\dot{\theta}_0 + 3\frac{\dot{V}}{V}\theta_0 = \frac{2}{\theta_0}.$$

The system of the second approximation:

$$\theta = \frac{1}{1-u}[\theta_0(1-2u) + \theta_1 u]; \quad \frac{1}{\theta} = (1-u)\left[(1-2u)\frac{1}{\theta_0} + \frac{1}{\theta_1}4u\right];$$

$$\dot{\theta}_0 + \frac{\dot{V}}{V}(9\theta_0 + 7\theta_1) = \frac{34}{\theta_0} - \frac{32}{\theta_1};$$

$$\dot{\theta}_1 + \frac{\dot{V}}{V}(4\theta_0 + 6\theta_1) = \frac{20}{\theta_0} - \frac{16}{\theta_1}.$$

/219

The system of the third approximation:

$$\theta = \frac{1}{1-u}\left[\theta_0\left(1 - \frac{9}{2}u + \frac{9}{2}u^2\right) + \theta_1(4u - 6u^2) + \theta_2\frac{1}{2}(-u + 3u^2)\right];$$

$$\frac{1}{\theta} = (1-u)\left[\frac{1}{\theta_0}\left(1 - \frac{9}{2}u + \frac{9}{2}u^2\right) + \frac{9}{2\theta_1}(2u - 3u^2) + \frac{9}{2\theta_2}(-u + 3u^2)\right];$$

$$\dot{\theta}_0 + \frac{\dot{V}}{V}\left(\frac{67}{2}\theta_0 + 40\theta_1 - \frac{7}{2}\theta_2\right) = 225\frac{1}{\theta_0} - \frac{234}{\theta_1} + \frac{9}{\theta_2};$$

$$\dot{\theta}_1 + \frac{\dot{V}}{V}\left(\frac{67}{12}\theta_0 + \frac{28}{3}\theta_1 + \frac{13}{12}\theta_2\right) = 39\frac{1}{\theta_0} - \frac{51}{2\theta_1} - 12\frac{1}{\theta_2};$$

$$\dot{\theta}_2 + \frac{\dot{V}}{V}\left(-\frac{83}{6}\theta_0 + \frac{31}{6}\theta_2\right) = -\frac{99}{\theta_0} + \frac{120}{\theta_1} - \frac{15}{\theta_2}$$

(the dot denotes differentiation with respect to ξ).

The question of initial conditions for the approximating system of ordinary differential equations remains open.

The start of the boundary layer ($\xi = 0$) is a singular point of equations of the approximating system. In fact, in the case of V_{cx}^{m} we can write

$$V = c^{\frac{1}{m+1}} (m+1)^{\frac{m}{m+1}} \xi^{\frac{m}{m+1}},$$

so that

$$\frac{\dot{V}}{V} = \frac{\beta}{2\xi}, \quad \text{where } \beta = \frac{2m}{m+1}.$$

In order to use any of the numerical integration methods one must have initial conditions in a point far removed from the singular point. It is easy to show that the solution of the approximating system of equations in any approximation can be represented in the form

$$\theta_k = A_k \sqrt{\xi}.$$

This can be easily shown by direct substitution into the system of equations. For example, in the second approximation

$$\frac{A_0}{2\sqrt{\xi}} + \frac{\beta}{2\xi}(9A_0\sqrt{\xi} + 7A_1\sqrt{\xi}) = \frac{34}{A_0\sqrt{\xi}} - \frac{32}{A_1\sqrt{\xi}};$$

$$\frac{A_1}{2\sqrt{\xi}} + \frac{\beta}{2\xi}(4A_0\sqrt{\xi} + 6A_1\sqrt{\xi}) = \frac{20}{A_0\sqrt{\xi}} - \frac{16}{A_1\sqrt{\xi}} \qquad /22$$

or

$$(1+9\beta)A_0 + 7\beta A_1 = \frac{68}{A_0} - \frac{64}{A_1};$$

$$4\beta A_0 + (1+6\beta)A_1 = \frac{40}{A_0} - \frac{32}{A_1}.$$

We note that quantities A_k are related to function φ of the exact Falkner-Skan differential equation by the following relationship; in particular, for A_0 we have the relation

$$\frac{1}{A_0} = \frac{\varphi''(0)}{\sqrt{2}}.$$

This expression can be used for estimating the accuracy of approximate solutions obtained with the method under consideration (see the table).

TABLE OF $\frac{1}{A_0} = \lim_{\xi \to 0} \frac{\sqrt{\xi}}{\theta_0}$

β	Approximation				Exact solution
	1st	2nd	3rd	4th	
-0.10	0,41833	separat'n	0,23246	0.22255	0,22576
0,00	0,50000	0,31692	0,32968	0,33191	0,33206
1,00	1,0000	0.87247	0,87056	0.87164	0,87157
2.00	1,32288	1,19371	1,19252	1,19321	1,19304

Once the value of θ_{κ} is found, the determination of the boundary-layer variables C_f, δ^* and δ^{**} is elementary:

$$C_f = \frac{2\tau_0}{\rho V^2} = \frac{2\sqrt{\nu}}{\theta_0};$$

$$\sqrt{\mathrm{Re}}\frac{V\delta^*}{l} \approx \theta_0 \approx \frac{1}{2}\theta_1 \approx \frac{1}{4}(\theta_0 + \theta_2) \approx \frac{\theta_1}{2} - \frac{\theta_2}{6} + \frac{\theta_3}{6};$$

$$\sqrt{\mathrm{Re}}\frac{V}{l}\delta^{**} \approx \frac{1}{2}\theta_0 \approx \frac{1}{3}\theta_1 - \frac{1}{6}\theta_0 \approx \frac{1}{8}\theta_0 - \frac{1}{6}\theta_1 +$$

$$+ \frac{5}{24}\theta_2 \approx \frac{1}{90}(-7\theta_0 - 27\theta_1 - 18\theta_2 + 13\theta_3),$$

respectively in the 1st, 2nd, 3rd and 4th approximations. Here l and V are the characteristic lengths and velocity /221

$$\sqrt{\mathrm{Re}} = \sqrt{Vl/\nu}.$$

In the vicinity of the point of boundary-layer separation obviously $\theta \to \infty$. In this vicinity the following approximating expression should be considered for θ

$$\theta \approx \frac{1}{1 - u\sqrt{u}}(c_0 + c_1 u + \ldots + c_{N-2}u^{N-2}).$$

The computation can be carried out with the old approximations, but for sufficiently high θ_n one should convert to the new functions

$$v_n = 1/\theta_n.$$

Numerical experiments carried out with the above substitution point to the possibility of computing up to the separation point and of finding the latter with sufficient accuracy.

The above method of generalized differential equations was applied successfully to calculation of boundary layers in a compressible heat-conducting gas [2] and to calculation of the compressible boundary layer of an ablating surface with allowance for diffusion in the gas mixture.

2. The Method of Finite Differences

Equations of the boundary layer are of the parabolic type and hence they can be treated to some extent by the tool of a priori estimates of the rate of convergence of difference methods presented in A. A. Samarskiy's method. Since he discussed model equations of heat conduction with simplest boundary conditions, we shall consider some of the specific aspects of the use of finite difference methods for problems of the boundary layer with incorporation of actual boundary conditions.

A large number of various difference schemes for solving boundary-layer equations is presently available. We shall consider only two of them, which are most interesting from this author's point of view. It was previously mentioned that the construction of the difference scheme depends on the kind of starting differential equations. In conjunction with this I selected two approaches to the construction of difference schemes of computations, the substance of which is determined to a large /22
extent by the form of the starting equations.

1. The first difference scheme considered will be that suggested by A. L. Dyshko [3]. If the independent variables x, y are replaced by von Mises variables, i.e., (x, ψ), where ψ is a stream function defined by the continuity equation, then the system of equations of an incompressible laminar boundary layer will be reduced to a single equation, which is the quasilinear equation of heat conduction

$$\frac{\partial z}{\partial x} = \nu \sqrt{U^2(x) - z}\, \frac{\partial^2 z}{\partial \psi^2}, \qquad (2.1)$$

where $z = U^2(x) - u^2(x, y)$, with the boundary conditions:

$$z(0, x) = U^2(x) > 0; \quad z(\infty, x) = 0; \quad z(\psi, 0) = 0.$$

Equation (2.1) has a singularity at the body's surface $\psi = 0$, since $\frac{\partial^2 z}{\partial \psi^2} \to \infty$ as $\psi \to 0$. Obviously, it is difficult to construct a regular method for computing this equation, firstly, due to the singularity in the equation and secondly, due to the need to solve the equation in the infinite region $0 \leq \psi < \infty$. The domain of the solution can be bounded by introducing new independent variables. In fact, let

$$X = \frac{\psi}{\psi_0 + \psi}; \quad V = (1 - X) z; \quad t = \frac{x}{\psi_0^2},$$

where ψ_0 is some constant allowing one to select a scale for ψ, then Eq. (2.1) will be

written in the form

$$\frac{\partial V}{\partial t} = \nu\sqrt{(1-X)W^2(t) - V(1-X)^{7/2}}\,\frac{\partial^2 V}{\partial X^2} \qquad (2.2)$$

with the boundary conditions:

$$V(0,t) = W^2(t) > 0; \quad V(1,t) = 0; \quad V(X,0) = 0,$$

where $W^2(t) = U^2(\varphi_0^2 x)$. For this equation, as before $\frac{\partial^2 V}{\partial X^2} \to \infty$ as $X \to 0$.

Now we must examine the nature of the singularity in the solution of Eq. (2.2) in order to be able to construct a difference scheme allowing for computation up to the body's surface (X = 0). We assume that: 1) $\frac{\partial V}{\partial t}$ is a continuous function of X in vicinity X = 0; 2) there exists $\left.\frac{\partial V}{\partial X}\right|_{\lambda=0} = A$; 3) boundary function $W^2(t) > 0$ is a monotonically increasing function. /223

The last limitation is not of substance and is used only to prove that

$$-\left.\frac{\partial V}{\partial X}\right|_{(0,t)} - W^2(t) > 0.$$

It is subsequently assumed that $\nu = 1$, which obviously, does not detract from generality, since it is reasonable to assume $\nu = \text{const}$ for an incompressible boundary layer.

We consider the nature of the solution of Eq. (2.2) in the vicinity of X = 0 for $t = t_0$. Let $W^2(t_0) = W_0^2$, and $\frac{\partial V}{\partial t} = p(X) > 0$. We write $q(X) = \frac{p(X)}{(1-X)^{7/2}}$. For $t = t_0$ Eq. (2.2) becomes the ordinary equation

$$q(X) = \sqrt{W_0^2(1-X) - V}\cdot V'', \qquad (2.3)$$

where $V'' = \left.\frac{d^2 V}{d\psi^2}\right|_{t=t_0}$

We introduce a new function

$$W = u - W_0^2 - AX. \qquad (2.4)$$

using which Eq. (2.3) is written as

$$q(X) = \sqrt{RX - W} \cdot W'',$$

where $R = -A - W_0^2$. Because of Eq. (2.4), the boundary conditions for W (X) will be

$$W(0) = W'(0) = 0. \tag{2.5}$$

According to the first assumption q(X) is a positive and continuous function in the vicinity of X = 0, when $W'' > 0$ in the same vicinity. With reference to Eq. (2.5) we find that $W(X) > 0$., from which it follows that $R > 0$. Then using the Cauchy formula

$$f(x) = \frac{1}{k!} \int_0^x (x-\tau)^k f^{(k+1)}(\tau) d\tau, \quad f^{(i)}(0) = 0, \quad i = 1, 2, \ldots, k,$$

and Eq. (2.5), we get for W(X)

$$W(X) = \int_0^X \frac{(X-\tau) q(\tau)}{\sqrt{R\tau - W(\tau)}} d\tau; \qquad /22$$

using the mean-value theorem, we may write

$$W(X) = \frac{q(\xi)}{\sqrt{R}} \int_0^X \frac{(X-\tau)}{\sqrt{1 - \frac{W}{R\tau}} \cdot \sqrt{\tau}} \cdot d\tau \tag{2.6}$$

and $0 < \xi < X$.

Evidently, we have the following representation in the vicinity of X = 0

$$\frac{1}{\sqrt{1 - \frac{W}{R\tau}}} = 1 + \frac{W}{2R\tau} + 0\left(\frac{W}{\tau}\right).$$

Substituting this expansion in Eq. (2.6), we get

$$W(X) = \frac{q(\xi)}{\sqrt{R}} \left[\frac{4}{3} X^{3/2} + \frac{1}{3R} X^2 + 0(X^2)\right];$$

Returning to function V(X) we get the following expression for the latter in the vicinity of X = 0:

$$V = a + bX + cX^{3/2} + dX^2 + 0(X^2),$$

where a, b, c and d are some constants.

If then the solution is approximated at some grid region in the form of parabolas $V_m = a_m + b_m X_m + c_m X_m^{3/2}$ and we write the second derivative as

$$V'' = \frac{\partial^2 V}{\partial X^2} \approx \frac{3}{4} c_m X_m^{-1/2},$$

then the first derivative $\frac{\partial V}{\partial t}$ can be written in the form

$$\frac{\partial V}{\partial t} \approx \frac{V_m^{n+1} - V_m^n}{\tau},$$

where τ is the lattice spacing in the direction of variable t and the difference scheme obtained as a result can be used for computation up to the body's surface X = 0.

The computational grid is specified by the equations /225

$$X_m = \frac{m^{4/3}}{M^{4/3}}, \quad t_n = n\tau,$$

where a nonuniform grid, becoming denser at the body's surface, is selected in the X direction. The selection of this kind of subdivision in the X direction was directed by two considerations: 1) the existence of a singularity in the solution's behavior in the vicinity of X = 0; 2) convenience of obtaining with this grid estimates of the approximation and stability of the difference systems. The mathematical manipulations are not given here, since they are quite simple but very cumbersome, and only the principal results will be given.

The principal term of the error of approximating the second derivative with respect to X can be written in the form

$$\Delta = \frac{3}{4} c_m (X_m^2) \cdot X_m^{-1/2} - 2 = 0(m^{-2}).$$

The approximating error of the first derivative with respect to time obviously is

$$\frac{V_m^{n+1} - V_m^n}{\tau} - \left.\frac{\partial V}{\partial t}\right|_{m,\,n} = 0(\tau).$$

It is known from the preceding lectures that the selection of an explicit difference scheme requires checking the latter for stability and determining the lattice spacing ratio which ensures this stability. It can be seen by examining the behavior of the linear part of the errors that the sufficient condition for stability of the difference scheme has the form

$$\tau = 0\left(\frac{1}{M^2}\right).$$

Then, if δ is used to denote the linear part of the error in solving the difference equation $\delta = u - u_{m,\,n}$, where u is the solution of the differential equation and $u_{m,n}$ is the solution of the corresponding difference equation), then it can be shown that

$$|\delta| \leq 0\left(\frac{1}{M^{\frac{2}{3}}}\right).$$

Apparently, this estimate of the convergence of the difference to the differential solution is somewhat on the high side.

One of the advantages of the above method is the small machine memory needed for computations. /2

2. In conclusion I wish to consider still another finite-difference method for solving equations of the laminar boundary layer [4]. The item of importance in writing the difference scheme in this method is the fact that the starting boundary-layer equations are written in the form of laws of conservation of mass, momentum and energy, or in the so-called divergent form.

The system of equations of the laminar boundary layer can then be written in the following vectorial form:

$$\iiint_V \operatorname{div} \vec{\Psi}_k \, dv = 0 \qquad (k = 1, 2, 3, \ldots, n). \tag{2.7}$$

Vectors $\vec{\Psi}_k$ express vector fluxes of mass, momentum and energy of a unit gas volume in the boundary layer. By virtue of arbitrariness of the volume V under consideration, Eqs. (2.7) yield the known differential equations of the boundary layer.

We restrict ourselves to a two-dimensional boundary layer. If the body's generatrix s and normal n to the body are taken as the coordinates, then the boundary

conditions take the form

$$\vec{\Psi}_{k\cdot\vec{n}}(\infty) = 0, \quad \vec{\Psi}_{k\cdot\vec{s}}(\infty) = \Phi_k(s),$$

where $\Phi_k(s)$ are known functions, obtained from solving the problem of external flow of an ideal gas past a body

$$\vec{\Psi}^{+}_{k\cdot\vec{n}}(0) = \vec{\Psi}^{-}_{k\cdot\vec{n}}(0),$$

where the plus and minus signs pertain to variables determined at both sides of the body's surface, which serves in boundary layer equations as a discontinuity surface. In the case when the body's surface disintegrates (evaporates or ablates), the conditions at it should be supplemented by the kinetic condition describing the evaporation or ablation mechanism. This condition is substantially nonlinear and has a marked effect on the convergence of the numerical method.

The idea of constructing difference representations of equations written in the form of integral conservation laws (2.7) is related to the following obvious equation

$$\iiint_V \operatorname{div} \vec{\Psi}_k \, dv \equiv \iint_s \vec{\Psi}_k \, d\vec{s} = 0$$

or in the two-dimensional case /227

$$\iint_s \operatorname{div} \vec{\Psi}_k \, ds \equiv \oint_L \vec{\Psi}_k \, d\vec{e} = 0, \tag{2.8}$$

where the directions of vectors $d\vec{s}$ and $d\vec{e}$ coincide respectively with the directions of normals to a surface element and to a contour element.

If now the domain of definition is subdivided by a curvilinear orthogonal grid $s_i = i\Delta s$, $n_j = j\Delta n$ and the integral $\int_L \vec{\Psi}_k d\vec{e}$ for the contour enclosing an area element with vertices $(s_i, n_{j-1}; s_i, n_{j+1}; s_{i+1}, n_{j+1}; s_{i+1}, n_{j-1})$ is written using some formulas of numerical integration, then we get as a result a difference equation corresponding to Eq. (2.8) for the area element in question. This method of constructing difference equations is a generalization of the known construction of difference equations of through computation for a heat conduction equation with a discontinuous coefficient suggested by A. A. Samarskiy and A. N. Tikhonov. We shall not consider in detail the difference equations obtained, since this was done by A. A. Samarskiy. It will be only noted that the difference scheme thus constructed can be used for

computations also in the case when the boundary-layer equations contain discontinuous coefficients.

The difference equations obtained by means of the above procedure can be solved by iterations. Here a solution of a system of linear equations must be found in each such iteration. In general the solution of the system of linear equations corresponding to the starting boundary layer equations could have been found using the method of matrix factorization. However, using the physical nature of vectors $\vec{\Psi}_k$ as vector fluxes of independent flow variables, it is possible to reduce the matrix factorization to a sequence of linear sweeps.

Vector $\vec{\Psi}_k$ can be written in the form

$$\vec{\Psi}_k = \vec{m}\varphi_k + \vec{f}_k \qquad (k = 1, 2, \ldots, n),$$

where $\vec{m}$ is the vector of the total mass flux; φ_k takes on values c_α, u and T, $\vec{f}_k$ is a vector whose projections are homogeneous functions of derivatives $\frac{\partial \varphi_k}{\partial n}$. If the iterations are organized so that only quantities φ_k are iterated with respect to the corrected quantities $\vec{m}$ and $\vec{f}_k$, then obviously with this organization of iterations the computation in each iteration reduces to sequential solution of the equations of motion, diffusion and conservation of energy by the sweep method. It is possible to carry out iterations also for terms $\vec{f}_k$, while retaining the linear sweep [5]. The advantage of this computation consists additionally in the fact that vector $\vec{m}$ with components ρu and ρv is sufficiently conservative, i.e., it changes little in the boundary layer. Hence the iteration process will converge sufficiently rapidly if conditions of impermeability or of a given gas penetration are satisfied at the body surface, i.e., if $\vec{m} \cdot \vec{n} = 0$ or $\vec{m} \cdot \vec{n} = f(s)$, where f(s) is a given function, for n = 0. If, however, $\vec{m} \cdot \vec{n}$ is an unknown quantity, and is to be found in the course of solution, which is the case for a boundary layer at a disintegrating surface, then the convergence of iterations may become markedly poorer or there may be no convergence at all. In this case the computations can be stabilized by using so-called damping of the quantity $(\vec{m} \cdot \vec{n})_0 = (\rho v)_0$ Quantity ρv in the $(l + 1)$th iteration is calculated from the formula

$$(\rho v)_{0\,l+1} = (\rho v)_{0\,l} + \delta[(\overline{\rho v})_0^{l+1} - (\rho v)_0^{l}],$$

where $(\overline{\rho v})_0^{l+1}$ is found from the evaporation condition at the surface.

The above approach was used for solving boundary-layer problems at ablating blunt-nosed bodies of revolution under various evaporation conditions at their surfaces. Computations were made for nonequilibrium evaporation, constant penetration rate and penetration rate sepcified in the form of a discontinuous function [4].

The same method was used for calculating the boundary layer in hypersonic nozzles. Flows in such nozzles have high negative pressure gradients and a sharp increase in the boundary layer thickness. The use of damping in the entire flow field in the boundary layer within the framework of the above numerical method provided results of good accuracy with a small number (3-5) of iterations.

We note in conclusion that the method of a priori estimates of convergence and error of the numerical method developed by A. A. Samarskiy for particular classes /229
of parabolic-type equations cannot be used for problems considered here due to their pronounced nonlinearity. Hence the accuracy of the method was estimated a posteriori, by decreasing lattice spacings and increasing the number of iterations.

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